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Doubly Connected Geodetic Number on Operations in Graphs

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Abstract: In this study, we study the concept of doubly connected geodetic number of a graph. A set $S\subseteq V$ in a graph G is a Doubly Connected Geodetic Set [DCGS] if S is a geodetic set and both induced subgraphs <S> and <V-S> are connected. The minimum cardinality of a doubly connected geodetic set and it is denoted by g_{dc} (G) is called doubly connected geodetic number of a graph G. A doubly connected geodetic set of cardinality $g_{dc}(G)$ is called $g_{dc}(G)$-set. We determine the doubly connected geodetic number in cartesian product, strong product, join of two graphs.

Key words: Cartesian product, geodetic number, strong product, join, composition, minimum cardinality, connected geodetic, doubly connected

INTRODUCTION

A u-v path of length d(u,v) is called a u-v geodesic of G and for a nonempty subset S of V(G), $I[S] = \cup_{u,v\in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if I[S] = V[G] and a geodetic set of minimum cardinality is the geodetic number g (G). The geodetic number was introduced by Chartrand *et al.* (2002). Nonsplit geodetic number g_{ns} (G) of a graph was studied by Tejaswini and Goudar (2016) and is defined as follows. The set $S\subseteq V(G)$ is a nonsplit geodetic set in G if S is a geodetic set and <V(G-S)> is connected, nonsplit geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a nonsplit geodetic set of G. The connected geodetic number was studied by Santhakumaran *et al.* a connected geodetic set of G is a geodetic set S such that the subgraph G[S] induced by S is connected. The minimum cardinality of a connected geodetic set of G is the connected geodetic number and is denoted by $g_c(G)$. The split geodetic number was studied by Venkanagouda and Ashalatha. The set $S\subseteq V(G)$ is a split geodetic set in G if S is a geodetic set and <V-S> is disconnected.

A vertex V is an extreme vertex in a graph G, if the subgraph induced by its neighbours is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G.

For any undefined term in this study (Harary, 1969; Chartrand and Zhang, 2006). The following theorems are used in the sequel.

Theorem 1.1 (Chartrand *et al.*, 2002): For any cycle C_n of order $n\geq 3$:

$$g(C_n) = \begin{cases} 2 \text{ if n is even} \\ 3 \text{ if n is odd} \end{cases}$$

Theorem 1.2 (Chartrand *et al.*, 2002): Every geodetic set of a graph contains its extreme vertices.

Theorem 1.3 (Chardrand and Zhang, 2006): For any cycle of order C_n of order $n\geq 3$:

$$a_0(C_n) = \begin{cases} \frac{n}{2} \text{if n is even} \\ \frac{n+1}{2} \text{i is odd} \end{cases}$$

Theorem 1.4 (Tejaswini and Goudar, 2016): Let k_2 and $G = C_n$ be the graphs then:

$$g_{ns}(K_2\times G) = \begin{cases} 2 \text{ if n is even} \\ 3 \text{ if } n>5 \text{ is odd} \\ 4 \text{if } n=3 \end{cases}$$

Theorem 1.5 (Venkanagouda *et al.*): For any path P_n of order n:

$$g_s(K_{2Pn}) = \begin{cases} 2, \text{ for} n=2 \\ 3, \text{ for} n\geq 3 \end{cases}$$

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In this study, we study the doubly connected geodetic set on Cartesian product, strong product and join of two graphs.

Doubly connected geodetic number of a graph: A set $S \subseteq V$ in a graph G is a Doubly Connected Geodetic Set [DCGS] if S is a geodetic set and both induced subgraphs <S> and <V-S> are connected. The minimum cardinality of a doubly connected geodetic set and it is denoted by g_{dc} (G) is called doubly connected geodetic number of a graph G. A doubly connected geodetic set of cardinality g_{dc} (G) is called g_{dc}(G)-set.

MATERIALS AND METHODS

Results on cartesian product of two graphs

Definition 3.1: The Cartesian product of the graphs H_1 and H_2, written as H_{1H2} is the graph with vertex set $V(H_1)\times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in H_1 H_2 if and only if either $u_1 = v_1$ and $(u_2, v_2)\in E(H_2)$ or $u_2 = v_2$ and $(u_1, v_1)\in E(H_1)$.

Theorem 3.2: For the cycle C_n of order $n\geq 3$:

$$g_{dc}(K_{2Cn}) = \begin{cases} \frac{n}{2}+2 \text{ if n is even} \\ \frac{n+1}{2}+2 \text{ if n is odd} \end{cases}$$

Proof: Consider V $(K_1) = \{u_1, u_2\}$ and V $(C_n) = \{v_1, v_2, ..., v_n\}$ by the definition of Cartesian product K_{2Cn}, C_n has two copies G_1 and G_2 in K_{2Cn}. Let $V = \{(u_1v_1), (u_1v_2), (u_1v_3), \dots, (u_1v_n), (u_2\, v_1), (u_2\, v_2), \dots, (u_2\, v_n)\}$ be the vertices in K_{2Cn}. We discuss the following cases.

Case (i): Suppose n is even, then $S_1 = \{(u_1\, v_1), (u_2^{vn/2}{}_{+1})\}$ be the geodetic set of K_{2Cn}, where $(u_1\, v_1)$, $(u_2^{vn/2}{}_{+1})$ are the antipodal vertices of K_{2Cn}. Thus, $I[S_1] = V[K_2\times C_n]$. But the induced subgraph <S_1> is not connected. Let us consider $S = S_1\cup S_2$ where $S_2 = \{(u_1\, v_2, ..., u_1^{vn/2}, u_2^{vn/2})\}$. Clearly the induced subgraph <S> and <V-S> are connected. Therefore, $g_{dc}(K_{2Cn}) = |S| = n/2+2$.

Case (ii): Suppose n is odd, then $S_1 = \{(u_1\, v_1), (u_2^{vn+1/2}), (u_2^{vn+1/2}{}_{+1})\}$ be the geodetic set of K_{2Cn} where $(u_2^{vn+1/2})$, $(u_2^{vn+1/2}{}_{+1})$ are the antipodal to the vertex $(u_1\, v_1)$. Thus, $I[S_1] = V[K_2\, C_n]$. But the induced <S_1> is not connected. Let us consider $S = S_1\cup S_2$ where $S_1 = \{(u_1\, v_2, \dots, u_1^{vn+1/2}, u_2^{vn+1/2})\}$. Clearly the induced subgraphs <S> and <V-S> are connected. Therefore, g_{dc} $(K_{2Cn}) = |S| = n+1/2+2$.

Theorem 3.3: For any path P_n of order $n\geq 3$, g_{dc} $(K_{2Pn}) = n+1$.

Proof: Let G_1, G_2 be the two copies of $G = P_n$ in K_{2Pn}. Consider $U = \{u_1, u_2\}$ be the vertex set of G_1, $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G_2 and $W = \{(u_1v_1), (u_1v_2), \dots, (u_1v_n)\, (u_2v_1), (u_2v_2), \dots, (u_2v_n)\}$ are the vertices of K_{2Pn}. Let $S_1 = \{(u_1v_1), (u_2v_n)\}$ be the split geodetic set and are the antipodal vertices in K_{2Pn}. But the induced subgraph <S_1> is not connected. Consider $S = S_1\cup S_2$ where $S_2 = \{(u_1v_2), (u_1v_3), \dots, (u_1v_n)\}$. Clearly both induced subgraphs <S> and <W-S> are connected. Hence, $|S| = 2+n-1 = n+1$. It follows that $g_{dc}(K_{2Pn}) = n+1$.

Theorem 3.4: For any path P_n of order $n\geq 2$, $g_{dc}(P_nP_n) = 2n-1$.

Proof: Let $G_1, G_2, \dots, G_n$ be the n disjoint copies of P_n in P_nP_n and $W = \{(u_1v_1), (u_1v_2), \dots, (u_1v_n)\, (u_2v_1), (u_2v_2), \dots, (u_2v_n), \dots, (u_nv_1), (u_nv_2), \dots, (u_nv_n)\}$ is the vertices of P_nP_n. Let $S_1 = \{(u_1v_1), (u_nv_n)\}$ be the geodetic set and are the antipodal vertices of P_n P_n. But the induced subgraph <S_1> is not connected. Consider $S = S_1\cup S_2$, where $S_2 = \{(u_2v_1), (u_3v_1), \dots, (u_nv_1), (u_nv_2), \dots, (u_nv_{n-1})\}\subseteq V(P_nP_n)-S_1$. It is known that the induced subgraphs <S> and <W-S> are connected. Hence, |S| is the doubly connected geodetic set of P_nP_n. It follows that $g_{dc}(P_n\, P_n) = |S| = 2n-1$.

Results on strong product of two graphs

Definition 4.1: The strong product of graphs G_1 and G_2, denoted G_1G_2 has vertex set $V(G_1)\times V(G_2)$ where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent with respect to the strong product if, $x_1 = x_2$ and $y_1y_2\in E$ (G_2) or $y_1 = y_2$ and $x_1x_2\in E(G_1)$ or $x_1x_2\in E(G_1)$ and $y_1y_2\in E(G_2)$.

Theorem 4.2: Let P_{n1} and P_{n2} be the paths of order $n_1\geq 2$ and $n_2\geq 3$, then:

$$g_{dc}(P_{n1}\, P_{n2}) = \begin{cases} n_1+n_2 \text{ if } n_1 \text{ is even and } n_1\leq n_2 \\ n_1+n_2-1 \text{ if } n_1 \text{ is odd and } n_1\leq n_2 \end{cases}$$

Proof: Consider $G = P_{n1}\otimes P_{n2}$ be the graph formed from n_1 copies of P_{n2}. Let $V(P_{n1}) = \{u_1, u_2, \dots, u(n_1)\}$ and $V(P_{n2}) = \{v_1, v_2, \dots, v_{n2}\}$, then $|V(G)| = n_1n_1$. We have the following cases.

Case (i): Suppose n_1 is even and $n_1\leq n_2$. We have two subcases.

Subcase (i) : If n_1 n_2, then $S_1 = \{(u_1, v_1, u_{n1}, v_1, u_{n2}, v_{n2}, u_1, v_{n2})\}$ be the geodetic set of G. We observed that <S_1> is not connected and <$V(G)-S_1$> is connected which is not a doubly connected geodetic set of G. Consider $S = S_1\cup S_2$

is the doubly connected geodetic set of G where:

$$S_2 = \left\{\left\{(u_2, v_2), (u_2, v_{n_2}\text{-}1), ..., \left(\frac{un_1}{2}, \frac{un_1}{2}\right), \left(\frac{un_1}{2}, v_{n_2}+1\text{-}\frac{n_1}{2}\right), ..., \left(\frac{un_1}{2}+1, \frac{un_1}{2}\right), \left(\frac{un_1}{2}+1, v_{n_2}+1\text{-}\frac{n_1}{2}\right), ..., (u_{n_1}\text{-}1, v2), (u_{n_1}\text{-}1, v_{n_2}\text{-}1)\right\} \cup \left\{\left(\frac{un_1}{2}+1, \frac{un_1}{2}+1\right), ..., \left(\frac{un_1}{2}+1, v_{n_2}+1\text{-}\frac{n_1}{2}\right)\right\}\right\}$$

Thus, it follows that:

$$g_{dc}(P_{n_1} P_{n_2}) = |S| = n_1+n_2$$

Subcase (ii): If $n_1 = n_2$, then:

$$S_1 = \{(u_1, v_1), (u_{n_1}, v_1), (u_{n_1}, v_{n_2}), (u_1, v_{n_2})\}$$

Be the geodetic set of G. We observed that <S_1> is not connected and <V(G)-S_1> is connected. Consider $S = S_1 \cup S_2$ is the doubly connected geodetic set G where:

$$S_2 = \{\{(u_2, v_2), (u_3, v_3), ..., (u_{n1\text{-}1}, v_{n2\text{-}1})\} \cup \{(u_2, v_{n2\text{-}1}), ..., (u_{n_1\text{-}1}, v_2)\}\}$$

Clearly:

$$g_{dc}(P_{n_1} P_{n_2}) = |S| = |S_1|+|S_2| = 4+n_1+n_2\text{-}4 = n_1+n_2$$

Case (ii): Suppose n_1 is odd and $n_1 \leq n_2$. We have two subcases.

Subcase (i): If $n_1 < n_2$, then:

$$S_1 = \{(u_1, v_1), (u_{n_1}, v_1), (u_{n_1}, v_{n_2}), (u_1, v_{n_2})\}$$

Be the geodetic set of G. We observed that <S_1> is not connected and <V(G)-S_1> is connected which is not a doubly connected geodetic set of G. Consider $S = S_1 \cup S_2$ is the doubly connected geodetic set of G where:

$$S_2 = \left\{\left\{(u_2, v_2), (u_2, v_{n_2\text{-}1}), ..., \left(\frac{u_{n_1}+1}{2}, \frac{u_{n_1}+1}{2}\right), \left(\frac{u_{n_1}+1}{2}, v_{n_2}+1\text{-}\frac{n_1+1}{2}\right), ..., \{(u_{n_1}\text{-}1, v_2), (u_{n_1}\text{-}1,, v_{n_2}\text{-}1)\}\right\} \cup \left\{\left(\frac{u_{n_1}+1}{2}, v_{n2}\text{-}\frac{n_1+1}{2}\right)\right\}\right\}$$

Thus, it follows that:

$$g_{dc}(P_{n_1} P_{n_2}) = |S| = |S_1 \cup S_2| = n_1+n_2\text{-}1$$

Subcase (ii): If $n_1 = n_2$, then:

$$S_1 = \{(u_1, v_1), (u_{n_1}, v_1), (u_{n_1}, v_{n_2}), (u_1, v_{n_2})\}$$

Be the geodetic set of G. We observed that <S_1> is not connected and and <V(G)-S_1> is connected. Consider $S = S_1 \cup S_2$ is the doubly connected geodetic set G where:

$$S_2 = \{\{(u_2, v_2), (u_3, v_3), ..., (u_{n1\text{-}1}, v_{n2\text{-}1})\} \cup \{(u_2, v_{n2\text{-}1}), ...(u_{n_1}, v_2)\}\}$$

Clearly:

$$g_{dc}(P_{n_1} P_{n_2}) = |S| = |S_1|+|S_2| = 4+n_1+n_{2\text{-}5} = n_1+n_{2\text{-}1}$$

Theorem 4.3: For the cycle C_n of order n = 4:

$$g_{dc}(K_2 C_n) = \begin{cases} n+2 & \text{if n is even} \\ n+3 & \text{if n is odd} \end{cases}$$

Proof: Let G be the strong product of K_2 C_n with C_n= 4. Consider K_2: u_1, u_2 and C_n: $v_1, v_2, ..., v_n$ be the vertices of K_2, C_n, respectively. V ($K_2 C_n$) = {(u_1v_1), (u_1v_2), ..., (u_1v_n), (u_2v_1), (u_2v_2), ..., (u_2v_n)} = 2n, we have the following cases.

Case (i): Suppose n is even cycle. Let S_1 = {(u_1v_1), ($u_1^{vn1/2+1}$), (u_2 v_1), ($u^{vn1/2+1}$)} be the geodetic set of K_2 C_n. We observed that the induced subgraphs <S_1> and <V (G)-S^1> are not connected. Consider $S = S_1 \cup S_2$, where:

$$S_2 = \left\{(u_1 v_2), ..., \left(u_1 \frac{vn_1}{2}\right), (u_2 v_2), ..., \left(u_2 \frac{vn_1}{2}\right)\right\} \subseteq V(G)\text{-}S_1$$

Forms a doubly connected geodetic set of G with minimum cardinality. It implies that both induced subgraphs <S> and <V (G)-S> are connected. Hence, it follows that $g_{dc}(K_2 C_n) = |S| = |S_1+S_2| = 4+n/2\text{-}1+n/2\text{-}1 = n+2$.

Case (ii): Suppose n is odd cycle. Let S_1 = {(u_1 v_1), (u_1, vn+1/2), ($u_1^{vn+1/2}$), (u_2, v1), (u2 $^{vn+1/2}$), ($u_2^{vn+1/2+1}$)} be the geodetic set of K_2 C_n. But the induced subgraphs <S_1> and <V (G)-S_1> are not connected. Consider $S = S_1 \cup S_2$, where:

$$S_2 = \left\{(u_1 v_2), ..., \left(u_1, \frac{v_n+1}{2}-1\right), \left(u_1, \frac{v_n+1}{2}+1\right), (u_2, v_2), \left(u_2 \frac{v_n+1}{2}-1\right)\right\} \subseteq V(G)\text{-}S_1$$

Forms a doubly connected geodetic set of G with minimum cardinality. It implies that both induced subgraphs <S> and <V (G)-S> are connected. Clearly:

$$g_{dc}(K_2 C_n) = |S| = |S_1 + S_2| = 6 + \frac{n+1}{2} - 2\frac{n+1}{2} - 2 = n+3$$

Results on join of two graphs

Definition 5.1: The join of two graphs G and H, denoted by G+H, is the graph with:

$$V(G+H) = V(G) \cup V(H) \text{ and } E(G+H) = E(G) \cup E(H) \cup \{u,v: u \in V(G) \text{ and } v \in V(H)\}$$

Theorem 5.1: If P_{n_1} and P_{n_2} be the paths then:

$$g_{dc}(P_{n_1} + P_{n_1}) = \{(n_2+3)/2$$

Proof: Let P_{n_1} and P_{n_2} be the paths, then:

$$G = P_{n_1} + P_{n_2} \text{ and } V(G) = n_1 + n_2$$

We have following cases.

Case (i): Suppose $n_1 = 2$ and $n_2 \geq 3$, n_2 is odd. Consider:

$$P_{n_1} = \{u_1, u_2\} \text{ and } P_{n_2} = \{v_1, v_2, ...v_n\}$$

If n_2 is odd, then the geodetic set $S = \{v_1, v_3, ... v_n\}$ contains $n_2+1/2$ vertices. But the induced subgraph <S> is not connected. Consider:

$$S_1 = S\ \{u_i\} = \{v_1, v_3, ..., v_n, u_i\}, \text{ for any } i = 1, 2 \text{ and } u_i \in P_{n_1}$$

Be the doubly connected geodetic set of G, such that induced subgraphs <S_1> and <V-S_1> are connected. Hence:

$$gdc(P_{n_1} + P_{n_2}) = |S \cup \{u_1\}| = |S| + 1 = \frac{n_2+1}{2} + 1 = \frac{n_2+3}{2}$$

Case (ii): Suppose $n_1 = 2$ and $n_2 \geq 3$, n_2 is even. Consider:

$$P_{n_1} = \{u_1, u_2\} \text{ and } P_{n_2} = \{v_1, v_2, ..., v_n\}$$

If n_2 is even, then the geodetic set $S = \{v_1, v_3, ..., v_{n-1}, v_n\}$ contains $n_2/2+1$ vertices. But the induced subgraph <S> is not connected. Consider $S_1 = S \cup \{u_i\} = \{v_1, v_3, ..., v_{n-1}, v_n, u_i\}$, for any $i = 1, 2$ and $u_i \in P_{n1}$ be the doubly connected geodetic set of G, such that induced subgraphs <S_1> and <V-S_1> are connected. Hence:

$$g_{dc}(P_{n_1} + P_{n_2}) = |S \cup \{u_i\}| = |S| + 1 = \frac{n_2}{2} + 1 + 1 = \frac{n_2+4}{2}$$

Case (iii): Suppose $n_1 = 3$ and $n_2 = 3$, n_2 is odd. If:

$$P_{n_1} = \{u_1, u_2, u_3\} \text{ and } P_{n_2} = \{v_1, v_2, ..., v_n\}$$

Then, the geodetic set $S = \{u_1, u_3\} = 2$ vertices. But the induced subgraph <S> is not connected. Consider $S_1 = S \cup \{u_2\}$ be the doubly connected geodetic set of G such that induced subgraphs <S_1> and <V(G)-S_1> are connected. Hence, $S_1 = |S \cup \{u_2\}| = |S|+1 = 2+1 = 3$. Therefore:

$$g_{dc}(P_{n_1} + P_{n_2}) = 3$$

Case (iv): Suppose $n_1, n_2 \geq 4$, consider $P_{n_1} = \{u_1, u_2, ..., u_n\}$ and $P_{n_2} = \{v_1, v_2, ..., v_n\}$, then the geodetic set $S = \{u_1, u_n, v_1, v_n\} = 4$ vertices, be the doubly connected geodetic set of G. Clearly induced subgraphs <S> and <V-S> are connected. Therefore, $g_{dc}(P_{n_1} + P_{n_2}) = 4$.

Theorem 5.3: If P_m be the path of order $m \geq 2$ and C_n be the cycle of order $n \geq 4$, then:

$$g_{dc}(P_m + \text{that 0 the vertex} = \begin{cases} \frac{n}{2}+1 \text{ if } \alpha_0(C_n) < m, \text{ n is even} \\ \frac{n+1}{2}+1 \text{ if } \alpha_0(C_n) < m, \text{ n is odd} \\ m \text{ if } \alpha_0(C_n) \geq m \end{cases}$$

Proof: If P_m be the path of order $m \geq 2$ and C_n be the cycle of order $n \geq 4$, then $V(P_m+C_n) = V(P_m)+V(C_n)$, where $V(P_m) = \{u_1, u_2, \dots, u_m\}$ and $V(C_n) = \{v_1, v_2, ..., v_n\}$. We have following cases.

Case (I): Suppose C_n is even and $\alpha_0(C_n)<m$, then $g(P_m+C_n) = \alpha_0(C_n) = n/2$ by theorem 1.3. But the induced subgraph <S> is not connected. Consider $S_1 = S \cup \{u_i\}$ for any i = 1, 2 and $u_i \in P_m$ be the doubly connected geodetic set of G such that the induced subgraphs $\langle S_1\rangle$ and $\langle V\text{-}S_1\rangle$ are connected. Hence, $g_{dc}(P_m+C_n) = n/2+1$.

Case (ii): Suppose C_n is odd and $\alpha_0(C_n) < m$, then g $(P_m+C_n) = \alpha_0(C_n) = n+1/2$ by theorem 1.3. But the induced subgraph $\langle S\rangle$ is not connected. Consider $S_1 = S\cup\{u_i\}$ for any i = 1, 2 and $u_i \in P_m$ is a doubly connected geodetic set of G. Clearly, the induced subgraphs $\langle S_1\rangle$ and $\langle V\text{-}S_1\rangle$ are connected. Hence, $g_{dc}(P_m+C_n) = |S\cup\{u_i\}| = |S|+1 = n+1/2+1$.

Case (iii): Consider the graphs with $\alpha_0(C_n) = m$. We have following subcases.

Subcase (i): Suppose P_m is even, then the geodetic set $S = \{u_1, u_3, ..., u_{m-1}, u_m\}$ is not a doubly connected geodetic set. Because the induced subgraph $\langle S\rangle$ is not connected. Consider $S_1 = \{u_1, u_2, ..., u_{m-1}, u_m\}$ contains m vertices such that both the induced subgraphs $\langle S_1\rangle$ and $\langle V\text{-}S_1\rangle$ are connected. Hence S_1 is a doubly connected geodetic set. Therefore $g_{dc}(P_m+C_n) = m$.

Subcase (ii): Suppose P_m is odd, then the geodetic set $S = \{u_1, u_3, ..., u_m\}$ is not a doubly connected geodetic set. Because the induced subgraph $\langle S\rangle$ is not connected. Consider $S_1 = \{u_1, u_2,... u_m\}$ contains m vertices, clearly both the induced subgraphs $\langle S_1\rangle$ and $\langle V\text{-}S_1\rangle$ are connected. Hence, S_1 is a doubly connected geodetic set. Therefore, $g_{dc}(P_m+C_n) = m$.

Theorem 5.4: Let, G be a complete graph and $H = K_n\text{-}e$, then $g_{dc}(G+H) = g_{dc}(H) = 3$.

Proof: If $G = K_n$, $H = K_n - e$ and $V(G+H) = V(G)\cup V(H)$, then the geodetic set $g(G+H) = g(H) = 2$. But the induced subgraph $\langle S\rangle$ is not connected. Hence, S is not a doubly connected geodetic set. Let us consider $S_1 = S\cup\{x\} = 2+1 = 3 = g_{dc}(H)$ where $\Delta(x) = n-1$ and $x\in V(G)\cup V(H)$ be the doubly connected geodetic set. Clearly both the induced subgraphs $\langle S_1\rangle$ and $\langle V\text{-}S_1\rangle$ are connected. Hence, $g_{dc}(G+H) = g_{dc}(H) = 3$.

Theorem 5.5: Let, G and H be a connected graphs of order n and m, respectively, such that $\Delta(G) = n-1$ and $\Delta(H) = m-1$, then $g_{dc}(G+H) = \min\{g(H), g(G)\}+1$ where g(H) and g(G) are the geodetic sets of H and G, respectively.

Proof: Let, $a\in V(G)$ and $b\in V(H)$ such that $\deg G(a) = \Delta(G) = n-1$ and $\deg H(b) = \Delta(H) = m-1$, then $S = g(G+H) = \min\{g(H), g(G)\}$. Since, the induced subgraph $\langle S\rangle$ is not connected. Consider$_1$ $= S\cup\{a\}$ or $S\cup\{b\}$. Clearly both the induced subgraphs $\langle S_1\rangle$ and $\langle V\text{-}S_1\rangle$ are connected. Hence, $g_{dc}(G+H) = \min\{g(H), g(G)\}+1$.

Theorem 5.6: If C_n and C_m be the cycles order n, m≥4, respectively and n≥m, then:

$$g_{dc}(C_n+C_m) = \begin{cases} \frac{m+2}{2} & \text{if n is even} \\ \frac{m+3}{2} & \text{if n is odd} \end{cases}$$

Proof: Suppose C_n and C_m be the cycle of order n, m≥4, respectively and $V(C_n+C_m) = V(C_n)+V(C_m)$, where $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $V(C_m) = \{u_1, u_2, ..., u_m\}$. If n≥m, then, we have following possibilities.

Case (i): Let, m is even, then the geodetic set S = g $(C_n+C_m) = \alpha_0(C_m) = m/2$ by theorem 1.3. But the induced subgraph $\langle S\rangle$ is not connected. Hence, S is not a doubly connected geodetic set. Consider $S_1 = S\cup\{v_i\}$ for any I = 1, 2, ..., n and $v_i \in C_n$ Such that both the induced subgraphs $\langle S_1\rangle$ and $\langle V\text{-}S_1\rangle$ are connected. Hence, S_1 is a doubly connected geodetic set, hence, $|S_1| = m+2/2$. Therefore, $g_{dc}(C_n+C_m) = m+2/2$.

Case (ii): Let, m is odd, then the geodetic set S = g $(C_n+C_m) = \alpha_0(C_m) = m+1/2$ by theorem 1.3. But the induced subgraph $\langle S\rangle$ is not connected. Hence, S is not a doubly connected geodetic set. Consider $S_1 = S\cup v_i$ for any i = 1, 2, ..., n and $v_i \in C_n$. Clearly both the induced subgraphs $\langle S_1\rangle$ and $\langle V(C_n+C_m)\text{-}S_1\rangle$ are connected. Hence, S_1 is a doubly connected geodetic set. Thus, $g_{dc}(C_n+C_m) = |S_1| = |S\cup v_i| = (m+3)/2$.

RESULTS AND DISCUSSION

Definition 6.1: The composition of two graphs G and H, denoted by G[H] is the graph with $V(G[H]) = V(G)\times V(H)$ and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1v_1\in E(G)$ or $u_1 = v_1$ and $u_2v_2\in E(H)$.

Theorem 6.2: Let, C_n be the cycle of order n≥4 and K_2 be the complete graph of order n = 2, then:

$$g_{dc}(C_n[K_2]) = \begin{cases} n+2 & \text{if nis even} \\ n+2 & \text{if n is odd} \end{cases}$$

Proof: Suppose C_n be the cycle of order n≥4 and K_2 be the complete graph, $V(C_n) = \{v_1, v_2, \dots, v_n\}$, $V(K_2) = \{u_1, u_2\}$ and $V(C_n[K_2]) = 2n$, we have following cases.

Case (i): C_n is even cycle. Let S = {(u_1v_1), ($u_1v_{(n/2+1)}$), (u_2v_1), ($u_2$vn/2+1)} is a geodetic set for $C_n[K_2]$ with minimum cardinality but the induced subgraphs <S> and <V-S> are not connected. Consider S = {(u_1v_2), …, ($u_1^{vn/2}$), (u_2v_2), …, ($u_2^{vn/2}$)} with n-2 vertices, then S_1 = S∪S be the doubly connected geodetic set of $C_n[K_2]$. Here both induced subgraphs <S_1> and <V-S_1> are connected. Hence, $|S_1|$ = | S∪S'| = |S|+|S'| = 4+n-2 = n-2. Therefore, g_{dc} (C_n [K_2]) = n+2.

Case(ii): C_n is odd cycle. Let S = {(u_1v_1), ($u_1^{vn+1/2}$) ($u_1^{vn+1/2+1}$), (u_2v_1), ($u_2^{vn+1/2}$), ($u_2^{vn+1/2+1}$)} = 2 g(c_n) = g($C_n[K_2]$). But the induced subgraphs <S> and <V-S> are not connected. Consider S' = {(u_1v_2), …, ($u_1^{vn-1/2}$), (u_2v_2), …, ($u_2^{vn-1/2}$)} with n–3 vertices, then S_1 = S∪S be the doubly connected geodetic set of $C_n[K_2]$. Here both induced subgraphs <S_1> and <V-S_1> are connected. Hence, $|S_1|$ = |S∪S'| = |S|+|S'| = 6+n-3 = n+3. Therefore, $g_{dc}(C_n[K_2])$ = n+3.

Theorem 6.3: Let, P_m and P_n be the paths of m, n≥4 and m≥n, then:

Proof: Let P_m and P_n be the paths of order n, m = 4 and V (P_n [P_m]) = mn, where P_n = {v_1, v_2, ..., v_n} and P_m = {u_1, u_2, ..., u_n }, we have the following cases.

Case (i): If P_n and P_m are even path. Then the geodetic set S = {(v_1u_1), (v_1u_3), ..., (v_1u_{m-1}), (v_nu_1), (v_nu_3), ..., (v_nu_{m-1}), (v_nu_m)} is the geodetic set with m+2 vertices but the induced subgraph <S> is not connected.

Let S' = {(v_2u_1), (v_3u_1), ..., ($v_{n-1}u_1$)} be the set with n-2 vertices, then S_1 = S∪S' be the doubly connected geodetic set. Clearly both the induced subgraphs <S_1> and <V-S_1> are connected. Hence, S_1 = |S∪S'| = |S|+|S'| = m+2+n-2 = m+n. Hence, $g_{dc}(P_n[P_m])$ = m+n.

Case (ii): When P_n and P_m are odd path. Then S = {(v_1u_1), (v_1u_3), ..., (v_nu_1), (v_nu_3), ..., (v_nu_m)} is the geodetic set with m+1 vertices but the induced subgraph <S> is not connected. Let S' = {(v_2u_1), (v_3u_1), ..., ($v_{n-1}u_1$)} be the set with n-2 vertices then, S_1 = S∪S' be the doubly connected geodetic set. Clearly both the induced subgraphs <S_1> and <V -S_1> are connected. Hence, S_1 = |S∪S'| = |S|+|S'| = m+1+n-2 = m+n-1. Hence, $g_{dc}(P_n[P_m])$ = m+n-1.

CONCLUSION

In this study, we found the exact value of doubly connected geodetic number for join, composition, Cartesian and strong product of two graphs.

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