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Convergence Criteria of Noor-Iterative Process in Hadamard Manifolds

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Abstract: The aim of this study is to study the convergence of Noor iterative procedure (a three-step procedure) for non-expansive mappings on Hadamard manifolds. The result generalizes several comparable results in the framework of Hadamard manifolds.

Key words: Noor iterative procedure, non-expansive mapping, Hadamard manifolds, procedure, framework, convergence

INTRODUCTION

Most of the results of non-expansive mappings have been obtained in normed linear spaces. The asymptotic behaviour of non-expansive mapping is an active research area in nonlinear functional analysis. The most important analytical problem is the existence of fixed points for nonlinear mapping T, i.e., solutions of x = T(x). Banach contraction principle states that the sequence of Picard iterates $\{T^n(x)\}$ converges strongly to a fixed point of T for any $x \in X$, if T is a contraction defined on a complete metric space X, one can find a huge literature about fixed points with different type of mappings (Goebel and Kirk, 1990; Kirk and Sims, 2001).

Mann (1953) introduced the most general iterative formula for approximation of fixed points of non-expansive mapping which is called Krasnoselskii-Mann iterative procedure. This procedure has been extensively studied by many researchers (Xu, 2002; Falset *et al*., 2001; Ishikawa, 1976; Reich, 1979; Kim and Xu, 2005; Xu and Noor, 2002). Then, Halpern (1967) gave an iterative procedure for a fixed point in 1967. Further, Ishikawa (1974), iteration procedure for approximating fixed points in Hilbert space has been introduced by Ishikawa (1974). Tan and Xu (1993) showed weak and strong convergence of Ishikawa iterative procedure for non-expansive mappings.

Noor (2000) introduced a three-step iterative process and studied the approximate solution of variational inclusion in Hilbert spaces. Many researchers studied this iteration process to approximate fixed points for various classes of nonlinear operators (Xu, 2002; Khan and Hussain, 2008; Suantai, 2005; Cho *et al*., 2004). In many respects, it is observed that a three-step iterative process is better than a two-step and a one-step iterative process for finding numerical results under different conditions (Abbas and Nazir, 2014; Glowinski and Tallec, 1989; Haubruge *et al*., 1998; Cho *et al*., 2004). Thus, we found that it is important to study three-step iterative processes in solving various numerical problems in the field of pure and applied sciences.

Goebel and Reich (1984) studied the behaviour of the sequence of Picard iterates in hyperbolic metric spaces. Li *et al*. (2010) studied the Mann and Halpern iterative algorithms for non-expansive mappings on Hadamard manifolds, i.e., complete simply connected Riemannian manifolds of non-positive sectional curvature. Motivated by the results by Li *et al*. (2010), we studied the Ishikawa iteration procedure for approximating a fixed point of non-expansive mappings in Hadamard manifolds (Chugh *et al*., 2014).

MATERIALS AND METHODS

Preliminaries: Let $p \in M$, where M is a connected m-dimensional Riemannian manifold. A Riemannian manifold is a Riemannian metric $\langle ., . \rangle$ with the corresponding norm denoted by $\|.\|$. We denote the tangent space of M at p by T_pM. We define the length of a piecewise smooth curve, c: [a, b]¬M joining p-q (i.e., c(a) = p and c(b) = q), by using the metric as:

$$L(c) = \int_a^b \|c'(t)\| dt$$

Then, the Riemannian distance d(p, q) is defined to be the minimal length over the set of all such curves joining p to q which induces the original topology on M. Let c be a smooth curve and Δ be the Levi-Civita connection associated to (M, $\langle , \rangle$). A smooth vector field X along c is said to be parallel if $\nabla_c X = 0$. If c' is parallel, then c is a geodesic and here $\|c'\|$ is a constant. A geodesic joining p to q in M is said to be minimal

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geodesic if its length equals d(p, q). A geodesic triangle. $\Delta(p_1, p_2, p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3 and three minimal geodesic joining p_i to p_{i+1} with i = 1, 2, 3 (mod 3).

A Riemannian manifold is complete if for any $p\in M$, all geodesics emanating from p are defined for all $-\infty<t<\infty$. By the Hopf-Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimizing geodesic. Thus, M, d is a complete metric space and bounded closed subsets are compact.

Now, the exponential map $\exp_p$: $T_pM\rightarrow M$ at $p\in M$ is such that $\exp_p\upsilon = \gamma_v(1, p)$ for each $\upsilon\in T_pM$, where $\gamma(.) = \gamma\upsilon$ (., p) is the geodesic starting at p with velocity υ. Then, $\exp_p t\upsilon = \gamma_1(t, p)$ for each real number t (Eq. 1).

Definition (Eq. 1): By Sakai (1996), a complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard Manifold. Now, we present some basic results. We assume that M is a m-dimensional Hadamard manifold.

Proposition (Eq. 1): By Sakai (1996), let $p\in M$. Then, $\exp_p$: $T_pM\rightarrow M$ is a diffeomorphism and for any two points p, $q\in M$ there exists a unique normalized geodesic joining p-q which is in fact a minimal geodesic. This result shows that M has the topology and differential structure similar to R^m. Thus, Hadamard manifolds and Euclidean spaces have some similar geometrical properties.

Proposition (Eq. 2): By Sakai (1996) (comparison theorem for triangles). Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle. For each i = 1, 2, 3 (mod 3) by γ_i: $[0, l_i]\rightarrow M$ the geodesic joining p_i to p_{i+1} and set $l_i = L(\gamma_i)$, $\alpha_i = \angle(\gamma_i'(0)\text{-}\gamma'_{i-1}(l_{i-1}))$. Then:

$$\alpha_1+\alpha_2+\alpha_3\leq\pi \tag{1}$$

$$l_i^2+l_{i+1}^2-2l_il_{i+1}\cos\alpha_{i+1}\leq l_{i-1}^2 \tag{2}$$

In terms of the distance and the exponential map, the inequality (Eq. 2) can be rewritten as:

$$d^2(p_i, p_{i+1})+d^2(p_{i+1}, p_{i+2})-2\left(\exp_{pi+1}^{-1}p_i, \exp_{pi+1}^{-1}p_{i+2}\right)\leq d^2(p_{i-1}, p_i) \tag{3}$$

Since:

$$\left(\exp_{pi+1}^{-1}p_i, \exp_{pi+1}^{-1}p_{i+2}\right) = d(p_i, p_{i+1})\,d(p_{i+1}, p_{i+2})\cos\alpha_{i+1}$$

Proposition (Eq. 3): By Sakai (1996), a subset $K\subset M$ is said to be convex if for any two points p and q in K, the geodesic joining p-q is contained in K, i.e., if γ: $[a, b]\rightarrow M$ is a geodesic such that: $p = \gamma(a)$ and $q = \gamma(b)$, then $((1\text{-}t)a+t\ b)\varepsilon$. K for all $t\varepsilon[0, 1]$. From now K will denote a nonempty, closed and convex set in M.

A real valued function f defined on M is said to be convex if for any geodesic γ of M, the composition function $f\circ\gamma$: $\mathbb{R}\rightarrow\mathbb{R}$ is convex that is:

$$(f\circ\gamma)(t\,a+(1\text{-}t)b)\leq t\,(f\circ\gamma)(a)+(1\text{-}t)$$
$$(f\circ n)(b)\ \text{for any a, b}\in\mathbb{R}\ \text{and}\ 0\leq t\leq 1$$

Proposition (4): By Sakai (1996), let d: $M\times M\rightarrow\mathbb{R}$ be a distance function. Then, d is a convex function with respect to the product Riemannian metric, i.e., given any pair of geodesics γ_1: $[0, 1]\rightarrow M$ and γ_2: $[0, 1]\rightarrow M$ the following inequality holds for all $t\in[0, 1]$:

$$d(\gamma_1(t), \gamma_2(t))\leq(1-t)\,d(\gamma_1(0), \gamma_2(0))+td(\gamma_1(1), \gamma_2(1))$$

In particular for each $p\in M$, the function d (., p): $M\rightarrow\mathbb{R}$ is a convex function. Let P_K denote the projection onto K defined by:

$$P_K(p) = \{p_0\in K: d(p, p_0)\leq d(p, q)\ \text{for all}\ q\in K\},\ \text{for all}\ p\in M$$

Definition (Eq. 2): By Ferreira and Oliveira (2002), let X be a complete metric space and $F\subset X$ be a nonempty set. A sequence $\{x_n\}\subset X$ is called Fejer convergent to F if:

$$d(x_{n+1}, y)\leq d(x_n, y)\ \text{for all}\ y\in F\ \text{and}\ n\geq 0$$

Lemma (Eq. 1): By Ferreira and Oliveira (2002), let X be a complete metric space. If $\{x_n\}\subset X$ is a Fejer convergent to a nonempty set $F\subset X$, then $\{x_n\}$ is bounded. Moreover, if a cluster point x of $\{x_n\}$ belongs to F, then $\{x_n\}$ converges to x.

Definition (Eq. 3): By Noor (2000), let $x_0\in X$ be arbitrary. If the sequence $\{x_n\}_{n=0}^{\infty}$ satisfies the condition:

$$\begin{aligned} x_{n+1} &= \alpha_n x_n+(1-\alpha_n)Ty_n\\ y_n &= \beta_n x_n+(1-\beta_n)Tz_n\\ z_n &= \lambda_n x_n+(1-\lambda_n)Tx_n \end{aligned} \tag{4}$$

for n = 0, 1, 2, 3, ..., then this is called the Noor iteration, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are the sequences such that $0\leq\{\alpha_n\}$, $\{n\beta\}$, $\{\lambda n\}\leq 1$ for all positive integers n.

In the next study, we study the convergence of Noor iteration for non-expansive smappings in Hadamard manifold. The Noor iteration in Hadamard manifolds M is as follows:

$$x_{n+1} = \exp_{x_n}\left(\left(1-\alpha_n\right)\exp_{x_n}^{-1} T\left(y_n\right)\right)$$
$$y_n = \exp_{x_n}\left(\left(1-\beta_n\right)\exp_{x_n}^{-1} T\left(z_n\right)\right) \quad (5)$$
$$z_n = \exp_{x_n}\left(\left(1-\lambda_n\right)\exp_{x_n}^{-1} T\left(x_n\right)\right)$$

for all $n\geq 0$ where $0<\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}<1$ and satisfy the following condition:

$$\sum_{n=1}^{\infty}\alpha_n\left(1-\alpha_n\right)=\infty,\ \sum_{n=1}^{\infty}\beta_n\left(1-\beta_n\right)=\infty,\ \sum_{n=1}^{\infty}\lambda_n\left(1-\lambda_n\right)=\infty \quad (6)$$

RESULTS AND DISCUSSION

Theorem (Eq. 7): Let K be a closed convex subset of M and T: K→K a non-expansive mapping with F = Fix (T)≠ϕ. Suppose $x_0\in M$. M and let $\{x_n\}$ be the sequence generated by the algorithm (Eq. 5) and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}(0, 1)$ satisfy the condition (Eq. 6). Then $\{x_n\}$ converges to a fixed point of T.

Proof: We know that K is a closed convex subset of M, thus, K is a complete metric space. Using Lemma (Eq. 1), it is sufficient to prove that $\{x_n\}$ is Fejer convergent to F and that all cluster points of $\{x_n\}$ belong to F. Now we suppose that $n\leq 0$ and $p\in F$ be fixed and γ_1, γ_2 and γ_3 denote the geodesic joining x_n to T (y_n), y_n to T (z_n) and z_n to T(x_n). Then, $x_n+1 = \gamma_1(1-\alpha_n)$, $y_n = 2(1-\beta n)$ and $z_n = 3(1-\lambda_n)$. Now using the convexity of distance function and the non-expansivity of T, we have:

$$d\left(x_{n+1}, p\right) = d\left(\gamma_1\left(1-\alpha_n\right), p\right) \leq \alpha_n d\left(x_n, p\right) + \left(1-\alpha_n\right)d\left(Ty_n, p\right) \leq \alpha_n d\left(x_n, p\right) + \left(1-\alpha_n\right)d\left(y_n, p\right) \quad (7)$$

and:

$$d\left(y_n, p\right) = d\left(\gamma_2\left(1-\beta_n\right), p\right) \leq \beta_n d\left(x_n, p\right) + \left(1-\beta_n\right)d\left(Tz_n, p\right) \leq \beta_n d\left(x_n, p\right) + \left(1-\beta_n\right)d\left(z_n, p\right) \quad (8)$$

$$d\left(z_n, p\right) = d\left(\gamma_3\left(1-\lambda_n\right), p\right) \leq \lambda_n d\left(x_n, p\right) + \left(1-\lambda_n\right)d\left(Tx_n, p\right) \leq \lambda_n d\left(x_n, p\right) + \left(1-\lambda_n\right) \quad (9)$$
$$d\left(x_n, p\right) \Rightarrow d\left(z_n, p\right) \leq d\left(x_n, p\right)$$

By Eq. 7-9, we obtain:

$$d\left(x_{n+1}, p\right) \leq d\left(x_n, p\right)$$

Hence, x_n is a Fejer convergent to F. Let x be a cluster point of $\{x_n\}$. Then, there exists a subsequence $\{n_k\}$ of n such that $x_{n_k} \to x$. Next, we prove:

$$\lim_{n\to\infty} d\left(x_n, Tx_n\right) = 0 \quad (10)$$

For this, let $p\in F$ and $n\geq 0$. Let $\Delta(x_n, q, p)$ be the geodesic triangle with vertices x_n, $q = Ty_n$ and p. From Lemma (Eq. 9) there exists a comparison triangle $\Delta(x'_n, q', p')$ which conserves the length of edge. Also, we have $x_{n+1} = \gamma_1 1(1-\alpha_n)$. Set $x'_{n+1} = \alpha_n x'_n+(1-\alpha_n)Ty'_n = \alpha_n x'_n+(1-\alpha_n)q'$ as its comparison point. By Lemma (Eq. 11):

$$\begin{aligned} d^2\left(x_{n+1}, p\right) &\leq \| x'_{n+1}-p'\|^2 = \| \alpha_n\left(x'_n-p'\right)+\left(1-\alpha_n\right)\left(q'-p'\right)\|^2 = \alpha_n \| x'_n-p\|^2 +\left(1-\alpha_n\right)\| q'-p'\|^2-\alpha_n\left(1-\alpha_n\right)\| x'_n-q'\|^2 \\ &= \alpha_n d^2\left(x, p\right)+\left(1-\alpha_n\right)d^2\left(Ty_n, p\right)-\alpha_n\left(1-\alpha_n\right)d^2\left(x_n, Ty_n\right) \leq \alpha_n d^2\left(x, p\right)+\left(1-\alpha_n\right)d^2\left(y_n, p\right)-\alpha_n\left(1-\alpha_n\right) \\ &\quad d^2\left(x_n, Ty_n\right) \end{aligned} \quad (11)$$

Now, let $\Delta(x_n, l, p)$ the geodesic triangle with vertices x_n, $l = Tz_n$ and p. From Lemma (Eq. 9) there exists a comparison triangle $\Delta(x'_n, l', p')$ which conserves the length of edge. Also, we have $y_n = \gamma_2(1-\beta_n)$ and set $y' = \beta_n x'_n+(1-\beta_n)Tz'_n = \beta_n x'_n+(1-\beta_n)Tz'_n = \beta_n x'_n+(1-\beta_n)l'$ similarly, we can obtain:

$$\begin{aligned} d^2\left(y_n, p\right) &\leq \| y'_n-p'\|^2 = \beta_n\left(x'_n-p'\right)+\left(1-\beta_n\right)\left(1'-p'\right)\|^2 = \beta_n \| x'_n-p'\|^2+\left(1-\beta_n\right)\| 1'-p'\|^2- \\ &\beta_n\left(1-\beta_n\right)\| x'_n-1'\|^2 = \beta_n d^2\left(x_n, p\right)+\left(1-\beta_n\right)d^2 \\ &\left(Tz_n, p\right)-\beta_n\left(1-\beta_n\right)d^2\left(x_n, Tz_n\right) \leq \beta_n d^2\left(x_n, p\right)+ \\ &\left(1-\beta_n\right)d^2\left(z_n, p\right)-\beta_n\left(1-\beta_n\right)d^2\left(x_n, Tz_n\right) \end{aligned} \quad (12)$$

Now, let $\Delta(x_n, m, p)$ be the geodesic triangle with vertices x_n, $m = Tx_n$ and p. From Lemma (Eq. 9), there exists a comparison triangle $\Delta(x'_n, m', p')$ which conserves the length of edge. Also, we have $z_n = \gamma_3(1-\lambda_n)$ and set $y' = \lambda_n x'_n+(1-\lambda_n)Tx'_n = \lambda_n x'_n+(1-\lambda_n)l'$. similarly, we can obtain:

$$\begin{aligned} d^2\left(z_n, p\right) &\leq \| z'_n-p'\|^2 = \lambda_n\left(x'_n-p'\right)+\left(1-\lambda_n\right)\left(m'-p'\right)\|^2 = \lambda_n \| x'_n-p'\|^2+\left(1-\lambda_n\right)\| m'-p'\|^2- \\ &\lambda_n\left(1-\lambda_n\right)\| x'_n-m'\|^2 = \lambda_n d^2\left(x_n, p\right)+\left(1-\lambda_n\right)d^2 \\ &\left(Tx_n, p\right)-\lambda_n\left(1-\lambda_n\right)d^2\left(x_n, Tx_n\right) \leq \lambda_n d^2\left(x_n, p\right)+ \\ &\left(1-\lambda_n\right)d^2\left(x_n, p\right)-\lambda_n\left(1-\lambda_n\right)d^2\left(x_n, Tx_n\right) \leq \\ &d^2\left(x_n, p\right)-\lambda_n\left(1-\lambda_n\right)d^2\left(x_n, Tx_n\right) \end{aligned}$$

Combining (Eq. 12 and 13), we obtain:

$$d^2(y_n,p) \le \beta_n d^2(x_n,p)+(1-\beta_n)\left[d^2(x_n,p)-\lambda_n(1-\lambda_n)d^2(x_n,Tx_n)\right]-\beta_n(1-\beta_n)d^2(x_n,Tz_n) \le \beta_n d^2(x_n,p)+(1-\beta_n)d^2(x_n,p)-\lambda_n(1-\lambda_n)(1-\beta_n)d^2(x_n,Tx_n)-\beta_n(1-\beta_n)d^2(x_n,Tz_n) \le d^2(x_n,p)-\lambda_n(1-\lambda_n)(1-\beta_n)d^2(x_n,Tx_n)-\beta_n(1-\beta_n)d^2(x_n,Tz_n) \quad (14)$$

Combining (Eq. 11 and 14):

$$d^2(x_{n+1},p) \le \alpha_n d^2(x_n,p)+(1-\alpha_n)\left[d^2(x_n,p)-\lambda_n(1-\lambda_n)(1-\beta_n)d^2(x_n,Tx_n)-\beta_n(1-\beta_n)d^2(x_n,Tz_n)\right]-\alpha_n(1-\alpha_n)d^2(x_n,Ty_n) \le \alpha_n d^2(x_n,p)+(1-\alpha_n)d^2(x_n,p)-\lambda_n(1-\lambda_n)(1-\alpha_n)(1-\beta_n)d^2(x_n,Tx_n)-\beta_n(1-\beta_n)(1-\alpha_n)d^2(x_n,Tz_n)-\alpha_n(1-\alpha_n)d^2(x_n,Tz_n)-\alpha_n(1-\alpha_n)d^2(x_n,Ty_n) \le d^2(x_n,p)-\lambda_n(1-\lambda_n)(1-\alpha_n)(1-\beta_n)d^2(x_n,Tx_n)-\beta_n(1-\beta_n)(1-\alpha_n)d^2(x_n,Tz_n)-\alpha_n(1-\alpha_n)d^2(x_n,Ty_n)$$

it follows that:

$$\lambda_n(1-\lambda_n)(1-\alpha_n)(1-\beta_n)d^2(x_n,Tx_n)-\beta_n(1-\beta_n)(1-\alpha_n)d^2(x_n,Tz_n)-\alpha_n(1-\alpha_n)d^2(x_n,Ty_n) \le d^2(x_n,p)-d^2(x_{n+1},p) \quad (15)$$

and:

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)d^2(x_n,Ty_n)<\infty$$

$$\sum_{n=1}^{\infty}\beta_n(1-\alpha_n)(1-\beta_n)d^2(x_n,Tz_n)<\infty$$

$$\sum_{n=1}^{\infty}\lambda_n(1-\lambda_n)(1-\alpha_n)(1-\beta_n)d^2(x_n,Tx_n)<\infty$$

which implies that:

$$\liminf_{n\to\infty} d(x_n,Ty_n)=0 \quad \liminf_{n\to\infty} d(x_n,Tz_n)=0 \text{ and } \liminf_{n\to\infty} d(x_n,Tx_n)=0 \quad (16)$$

because otherwise $d(x_n, T(x_n))\ge a$, $d(x_n, T(y_n))\ge b$ and $d(x_n, T(z_n))\ge c$ for all $n\ge 0$ and for some $a, b, c>0$ and then:

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)d^2(x_n,Ty_n)\ge b\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$$

and:

$$\sum_{n=1}^{\infty}\beta_n(1-\alpha_n)(1-\beta_n)d^2(x_n,Tz_n)\ge c\sum_{n=1}^{\infty}\beta_n(1-\alpha_n)<\infty$$

and:

$$\sum_{n=1}^{\infty}\lambda_n(1-\lambda_n)(1-\alpha_n)(1-\beta_n)d^2(x_n,Tx_n)\ge a\sum_{n=1}^{\infty}\lambda_n(1-\alpha_n)(1-\beta_n)<\infty$$

which is a contradiction with Eq. 15. On the other hand, the non-expansivity of T and convexity of the distance function, implies that:

$$d(x_{n+1},T(x_{n+1})) \le d(x_{n+1},T(x_n))+d(T(x_n),T(x_{n+1})) \le d(x_{n+1},T(x_n))+d(x_n,x_{n+1}) \le \alpha_n d(x_n,T(x_n))+(1-\alpha_n)$$

$$d(T(y_n),T(x_n)) \le \alpha_n d(x_n,T(x_n))+(1-\alpha_n)d(T(y_n),x_n)+(1-\alpha_n)d(y_n,T(x_n))$$

now:

$$d(Ty_n,x_n) \le \beta_n d(Tz_n,x_n)+(1-\beta_n)d(Tz_n,x_n) \le \beta_n d(Tz_n,x_n)+(1-\beta_n)d(Tz_n,x_n) \le d(Tz_n,x_n)$$

and:

$$d(Tz_n,x_n) \le \lambda_n d(x_n,x_n)+(1-\lambda_n)d(Tx_n,x_n) \le \lambda_n d(x_n,x_n)+(1-\lambda_n)d(x_n,x_n) \le d(x_n,x_n)$$

Therefore:

$$d(x_{n+1},T(x_{n+1})) \le \alpha_n d(x_n,T(x_n))+(1-\alpha_n)d(x_n,T(x_n)) \le d(x_n,T(x_n))$$

This means that $\{d(x_n, T(x_n)\}$ is a monotone sequence. Combining this and (Eq. 15) we get that (Eq. 9) holds. Then, since:

$$d(x,T(x)) \le d(x,x_{n_k})+d(x_{n_k},T(x_{n_k}))+d(T(x_{n_k}),T(x)) \le 2d(x_{n_k},x)+d(x_{n_k},T(x_{n_k}))$$

by taking limit, we deduce that $d(x, T(x)) = 0$ which means that $x\in Fix(T)$. Now, we obtain some results which are the consequences of the above result.

Corollary (Eq. 8): Let K be a closed convex subset of M and T: K→K a non-expansive mapping with F = Fix(T) ≠ ∅. Let $x_0 \in M$ and let $\{x_n\}$ be the sequence generated by the algorithm:

$$x_{n+1} = \exp_{x_n}(1-\alpha_n)\exp_{x_n}^{-1} T(y_n)$$
$$y_n = \exp_{x_n}(1-\beta_n)\exp_{x_n}^{-1} T(x_n)$$

for all $n \geq 0$, where $0 < \{\alpha_n\}, \{\beta_n\} < 1$. Then, x_n converges to a fixed point of T.

Proof: To obtain the desired result put $\lambda_n = 0$ in theorem (Eq. 7).

Corollary (Eq. 9): Let K be a closed convex subset of M and T: K→K a non-expansive mapping with F = Fix(T)≠∅ suppose that $\{\alpha_n\} \subset (0, 1)$ satisfy the condition:

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$$

Let $x_0 \in M$ and let $\{x_n\}$ be the sequence generated by the algorithm:

$$x_{n+1} = \exp_{x_n}(1-\alpha_n)\exp_{x_n}^{-1} T(x_n) \text{ for all } n \geq 0$$

Then, $\{x_n\}$ converges to a fixed point of T.

Proof: To obtain the desired result put $\beta_n = \lambda_n = 0$ in theorem (Eq. 7).

CONCLUSION

In the recent years, some algorithms for solving variational inequalities and minimization problems have been extended from the Hilbert space framework to the more general setting of Riemannian manifolds. The convergence of Mann, Halpern and Ishikawa iterative procedures to a fixed point for non-expansive mappings on Hadamard manifolds has been studied (DoCarmo, 1992; Sakai, 1996).

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