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Power Series Solutions for Non-Linear Systems of Partial Differential Equations

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Abstract: In this study, we would like to discuss the analytical solution of non-linear partial differential systems. The solutions are obtained using the technique of power series to solve linear ordinary differential equations. This method ensures the theoretical exactness of the approximate solution. Several systems are solved using this method and comparisons of the approximate solutions with the exact ones are demonstrated.

Key words: Non-linear PDEs, power series method, analytical solutions, approximate, ensure, exact

INTRODUCTION

It is well known that there are several methods that can be used to find general solutions to linear PDEs. On the contrary for non-linear PDEs it is well known that there are no generally applicable methods to solve such nonlinear equations. A glance at the literature shows that there are some known methods which have been applied to solve special cases of nonlinear PDEs. For example, the split-step method is a computational method that has been used to solve specific equations like nonlinear Schrodinger equation (Taha and Ablowitz, 1984).

Nevertheless, some thechniques can be used to solve serveral types of nonlinear equations such as the homotopy principle which is the most powerful method to solve underdetermined equations. In some cases, a PDE can be solved via., perturbation analysis in which the solution is considered to be a correction to an equation with a known solution (Chun *et al*., 2009). Alternatively there are numerical techniques that solve nonlinear PDEs such as the finite difference method and the finite element methods (Pelosi, 2007). Many interesting problems in science and engineering can be solved in this way using computers.

A general approach to solve PDEs uses the symmetry property of differential equations, the continuous infinitesimal transformations of solutions to solutions (Lie theory) (Hawkins, 2000). The continuous group theory, Lie algebras and differential geometry are used to understand the structure of linear and non-linear partial differential equations. Then, generating integrable equations to find their Lax pairs recursion operators, Backlund transform and finally finding exact analytic solutions to PDEs (Polyanin and Zaitsev, 2004).

MATERIALS AND METHODS

Power series method for nonlinear partial differential equations: Power series is an old technique for solving linear ordinary differential equations. The efficiency of this standard technique in solving linear ODE with variable coefficients is well known. An extension known as Frobenius method allows tackling differential equations with coefficients that are not analytic. Recently the method has been used to solve non-linear ODEs (Fairen *et al*., 2008). Furthermore, Kurulay and Bayram (2009) used power series to solve linear second order PDEs.

In this research, we apply the power series method to nonlinear PDEs. Analytical solutions are found by using algebraic series. Manipulation of the equations leads to very convenient recurrence relations that ensure the exactness of the solution as well as the computational efficiency of the method. The method is straightforward and can be programmed using any mathematical package. The efficiency of this method is illustrated through some examples and obtained solutions compared with exact solutions.

The general algebra for solving nonlinear ODEs is explained by considering an analytical function $x = x(\tau)$ defined in $\{1: 0\leq\tau\leq1\}$. Assume its expansion in power series as:

$$x(\tau) = \sum_{k-0}^{\infty} a_{1k}\tau^k \quad (1)$$

And for any integer m, $x(\tau)$ of power m is expressed as:

$$x(\tau)^m = \sum_{k=0}^{\infty} a_{mk}\tau^k \quad (2)$$

The following relation is an essential condition to be satisfied in order to reveal the desired recurrence relation:

$$x^m(\tau) = x^{m-1}(\tau)x(\tau) \quad (3)$$

After replacing the series expressions in each factor of Eq. 3 the following recurrence relation is obtained:

$$a_{mk} = \sum_{p=0}^{k} a_{(m-1)p} a_{1(k-p)} = \sum_{p=0}^{k} a_{1p} a_{(m-1)(k-p)} \quad (4)$$

To find the series expansion of a product of two functions assumes that the functions $f(\tau)$ and $g(\tau)$ are analytic at $\tau = 0$ and are defined in $\{1: 0\leq\tau\leq1\}$. Let:

$$f(\tau) = \sum_{i=0}^{\infty} a_i \tau^i;\ g(\tau) = \sum_{i=0}^{\infty} b_i \tau^i \quad (5)$$

And:

$$h(\tau) = f(\tau)\, g(\tau) \quad (6)$$

Then, the function h is also analytic at $\tau = 0$ and defined in I, therefore the series expansion for $h(\tau)$ is:

$$h(\tau) = \sum_{i=0}^{\infty} c_i \tau^i \quad (7)$$

Where:

$$c_i = \sum_{s=0}^{i} a_s b_{i-s} = \sum_{s=0}^{i} a_{i-s} b_s;\ \ i = 0, 1, 2, \ldots \quad (8)$$

and the series expansion for the nth derivative $x(\tau)$ can be written as:

$$x^{(n)}(\tau) = \sum_{k=0}^{\infty} \varnothing_{nk}\, a_{(k+n)} \tau^k \quad (9)$$

where, $\varphi_{kj} = (j+1)(j+2)\ldots(j+k)$, where k and j are postive inttegers. To generalize this method for solving nonlinear PDEs, let u(x, y) be a function of two variables and suppose that it is analytic in the domain $G\subseteq R^2$ and assume that the point (x_0, y_0) in G. The function u(x, y) is then represented as:

$$u(x, y) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij} (x-x_0)^i (y-y_0)^j \quad (10)$$

To find the representation series for any power of u(x, y) a condition as in Eq. 3 will be applied:

$$\mathbf{u^m (x, y) = u^{m-1}(x, y) u(x, y)} \quad (11)$$

If the series expansion of $u^m(x, y)$ is written as:

$$u^{(m)}(x, y) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij}^{(m)} (x-x_0)^i (y-y_0)^j \quad (12)$$

Then, using the relation in Eq. 11, the coefficients of u^m expressed as:

$$a_{ij}^{(m)} = \sum_{s=0}^{j}\sum_{p=0}^{i} a_{ps}^{(m-1)}\, a_{(i-p)(j-s)}^{(1)} \quad (13)$$

A representation for any derivative of u with respect to x or y for any order and for any power of them can be found by generalization the equivalent relation for ODEs. Some examples will be used to explain the method.

RESULTS AND DISCUSSION

Numerical examples: To illustrate the technique and exactness of the approximate solution, we now investigate some examples of nonlinear PDEs in detail.

Example 1: The nonlinear diffusion equation is considered:

$$u_t = \frac{\partial}{\partial x}(u^m u_x) \quad (14)$$

where, m is a positive integer. Let m = 2, then the equation is:

$$u_t - \frac{\partial}{\partial x}\left(u^2 \frac{\partial u}{\partial x}\right) - u^2 u_{xx} + 2uu_x^2 \quad (15)$$

with initial condition:

$$u(x, 0) = \frac{x+h}{2\sqrt{c}} \quad (16)$$

where, c>0 and h is an arbitrary constant. The exact solution of the given equation is Chun *et al.* (2009):

$$u(x, t) = \frac{x+h}{2\sqrt{c-t}}$$

Assume the solution u (x, t) as a power series in x and t:

$$u(x, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} x^i t^j \quad (17)$$

By differentiating both sides of Eq. 17 with respect to x, we will get the expansion series of u_x and u_{xx}:

$$u_x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (i+1) a_{(i+1),\, j} x^i t^j \quad (18)$$

$$u_{xx} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (i+1)(i+2)\, a_{(i+2),\, j} x^i t^j \quad (19)$$

The power series of u^2, u^2u_{xx} and uu_x^2 are obtained by applying Eq. 11 and the above series:

$$u^2 = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left[\sum_{s=0}^{j}\sum_{t=0}^{i} a_{ts}\, a_{(i-t)(j-s)}\right] x^j t^j \quad (20)$$

$$u^2 u_{xx} \sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\left[\begin{array}{l}\sum_{q=0}^{j}\sum_{p=0}^{i}(i-p+1)(i-p+2)a_{(t-p+2)(j-q)}\\ \left(\sum_{s=0}^{q}\sum_{t=0}^{p} a_{ts}a_{(p-t)\,(q-s)}\right)\end{array}\right] x^i t^j \quad (21)$$

Substitute these series into Eq. 11 to obtain the recurrence relation:

$$uu_x^2 = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left[\begin{array}{l}\sum_{q=0}^{j}\sum_{p=0}^{i} a_{(i-p)\,(j-q)}\\ \left(\sum_{s=0}^{q}\sum_{t=0}^{p}(t+1)(p-t+1)a_{(t+1)s}a_{(p-t+1)(q-s)}\right)\end{array}\right] x^j t^j \quad (22)$$

$$a_{i(j+1)} = \frac{1}{j+1}\left[\sum_{q=0}^{j}\sum_{p=0}^{i}(i-p+1)(i-p+2)\, a_{(i-p+2)\,(j-q)} \left(\sum_{s=0}^{q}\sum_{t=0}^{p} a_{ts}\, a_{(p-t)\,(q-s)}\right) + 2\sum_{q=0}^{j}\sum_{p=0}^{i} a_{\,(i-p)\,(j-q)} \left(\sum_{s=0}^{q}\sum_{t=0}^{p}(t+1)(p-t+1)\, a_{(t+1)s}\, a_{(p-t+1)\,(q-s)}\right)\right] \quad (23)$$

Where:

$$a_{i,\,0} = \begin{cases}\frac{h}{2\sqrt{c}} & \text{if } i = 0\\ \frac{1}{2\sqrt{c}} & \text{if } i = 1\\ 0 & \text{o.w}\end{cases}$$

By applying the recurrence relations Eq. 23 for several values of i and j. The polynomial approximation for u(x, t) is obtained:

$$\tilde{u}\,(x,\,t) = \frac{h}{2\sqrt{c}} + \frac{ht}{4c^{3/2}} + \frac{3ht^2}{16c^{5/2}} + \frac{5ht^3}{32c^{7/2}} + \frac{x}{2\sqrt{c}} + \frac{tx}{4c^{3/2}} + \frac{3t^2x}{16c^{5/2}} + \frac{5t^3x}{32c^{7/2}} \quad (24)$$

Table 1 demonstrates the difference between the approximate solution and the exact one for several values of x and t when h = 1 and c = 10.

Example 2: Consider the nonlinear system:

$$\begin{aligned} u_t + vu_x + u - 1 &= 0\\ v_t + uv_x - v - 1 &= 0\end{aligned} \quad (25)$$

Subject to the initial conditions u (x, 0) = e^x, v(x, 0) = e^{-x}. The exact solution is u(x, t) = e^{x-t}, v(x, t) = e^{t-x} (Bataineh *et al*., 2008). In order to solve the given system using the power series method, the solutions u and v are considered as:

$$u\,(x,t) = \sum_{i-0}^{N}\sum_{j-0}^{N} a_{i,\,j}\; x^i t^j \quad (26)$$

$$v\,(x,t) = \sum_{i-0}^{N}\sum_{j-0}^{N} b_{i,\,j}\; x^i t^j \quad (27)$$

We use the representation of the solutions in Eq. 26 and 27 to write the power series expansion of the products vu_x and uv_x. Then, we obtain the recursion (Eq. 28):

$$a_{i,\,j+1} = \frac{1}{j+1}\left[\delta_{i,\,j,\,0} - a_{i,\,j} + \sum_{t=0}^{j}\sum_{s=0}^{i}(s+1)\, b_{i-s,\,j-t}\; a_{s+1,\,t}\right] \quad (28)$$

$$b_{i,\,j+1} = \frac{1}{j+1}\left[d_{i,\,j,\,0} - b_{i,\,j} + \sum_{t=0}^{j}\sum_{s=0}^{i}(s+1)\, a_{i-s,\,j-t}\; b_{s+1,\,t}\right] \quad (29)$$

Where:

$$\delta_{i,\,j,\,0} = \begin{cases}1 & \text{if } i = j = 0\\ 0 & \text{o.w}\end{cases}$$

After solving Eq. 28 and 29 for i = 0, ..., 3 and j = 0, …, 3 we obtain the polynomials:

$$\tilde{u}\,(x,t) = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + x - tx + \frac{t^2x}{2} - \frac{t^3x}{6} + \frac{x^2}{2} - \frac{tx^2}{2} + \frac{t^2x^2}{4} - \frac{t^3x^2}{12} + \frac{x^3}{6} - \frac{t^3x}{6} + \frac{t^2x^3}{12} - \frac{t^3x^3}{36} \quad (30)$$

And:

$$\tilde{v}\,(x,t) = 1 + t + \frac{t^2}{2} - \frac{t^3}{6} - x - tx - \frac{t^2x}{2} - \frac{t^3x}{6} + \frac{x^2}{2} + \frac{tx^2}{2} + \frac{t^2x^2}{4} + \frac{t^3x^2}{12} - \frac{x^3}{6} - \frac{t^3x}{6} - \frac{t^2x^3}{12} - \frac{t^3x^3}{36} \quad (31)$$

The difference between approximate solutions and the exact solutions for Eq. 25 are shown in Table 2 and 3.

Table 1: The difference between exact and approximate solutions for example 1

t	|u(0, t)-ũ(0, t)|	|u (0.2, t)-ũ(0.2, t)|	|u(0.4, t)-ũ(0.4, t)|	|u(0.6, t)-ũ(0.6, t)|	|u(0.8, t)-ũ(0.8, t)|	|u(1, t)-ũ(1, t)|
0	0	0	0	0	0	0
0.2	7.04×10^{-9}	8.45×10^{-9}	9.86×10^{-9}	1.13×10^{-8}	1.27×10^{-8}	1.41×10^{-8}
0.4	1.15×10^{-7}	1.38×10^{-7}	1.61×10^{-7}	1.84×10^{-7}	2.07×10^{-7}	2.30×10^{-7}
0.6	5.92×10^{-7}	7.11×10^{-7}	8.29×10^{-7}	9.48×10^{-7}	1.07×10^{-6}	1.18×10^{-6}
0.8	1.91×10^{-6}	2.29×10^{-6}	2.67×10^{-6}	3.05×10^{-6}	3.44×10^{-6}	3.82×10^{-6}
1.0	4.75×10^{-6}	5.70×10^{-6}	6.65×10^{-6}	7.60×10^{-6}	8.56×10^{-6}	9.50×10-6

Table 2: The difference between exact and approximate solutions of u (x, t) for example 2

t	|u(0, t)-ũ(0, t)|	|u (0.2, t)-ũ(0.2, t)|	|u(0.4, t)-ũ(0.4, t)|	|u(0.6, t)-ũ(0.6, t)|	|u(0.8, t)-ũ(0.8, t)|	|u(1, t)-ũ(1, t)|
0	0	2.61×10^{-9}	3.42×10^{-7}	6.00×10^{-6}	4.62×10^{-5}	2.26×10^{-4}
0.2	2.48×10^{-9}	8.94×10^{-9}	2.76×10^{-7}	4.91×10^{-6}	3.78×10^{-5}	1.85×10^{-4}
0.4	3.10×10^{-7}	3.76×10^{-7}	2.32×10^{-7}	3.46×10^{-6}	3.03×10^{-5}	1.51×10^{-4}
0.6	5.16×10^{-6}	6.31×10^{-6}	7.52×10^{-6}	6.12×10^{-6}	1.39×10^{-5}	1.10×10^{-4}
0.8	3.8×10^{-5}	4.62×10^{-5}	5.62×10^{-5}	6.62×10^{-5}	6.34×10^{-5}	1.04×10^{-4}
1.0	1.8×10^{-4}	2.2×10^{-4}	2.63×10^{-4}	3.19×10^{-4}	3.75×10^{-4}	3.95×10^{-4}

Table 3: The difference between exact and approximate solutions of v (x, t) for example 2

t	|u(0, t)-ũ(0, t)|	|u (0.2, t)-ũ(0.2, t)|	|u(0.4, t)-ũ(0.4, t)|	|u(0.6, t)-ũ(0.6, t)|	|u(0.8, t)-ũ(0.8, t)|	|u(1, t)-ũ(1, t)|
0	0	2.48×10^{-9}	3.10×10^{-7}	5.16×10^{-6}	3.78×10^{-5}	1.76×10^{-4}
0.2	2.60×10^{-9}	8.94×10^{-10}	3.76×10^{-7}	6.31×10^{-6}	4.62×10^{-5}	2.15×10^{-4}
0.4	3.42×10^{-7}	2.76×10^{-7}	2.32×10^{-7}	7.52×10^{-6}	5.62×10^{-5}	2.63×10^{-4}
0.6	6.00×10^{-6}	4.91×10^{-6}	3.46×10^{-6}	6.12×10^{-6}	6.62×10^{-5}	3.19×10^{-4}
0.8	4.61×10^{-5}	3.78×10^{-5}	3.03×10^{-5}	1.38×10^{-5}	6.34×10^{-5}	3.75×10^{-4}
1.0	2.26×10^{-4}	1.85×10^{-4}	1.51×10^{-4}	1.10×10^{-4}	1.05×10^{-4}	3.95×10^{-4}

CONCLUSION

The method has been successfully applied directly to some examples of nonlinear PDEs without using linearization, perturbation or restrictive assumptions. It provides the solution in terms of convergent series with easily computable components and the results have shown remarkable performance. The efficiency of this method has been demonstrated by solving nonlinear PDEs and systems of nonlinear PDEs. A comparison of this method with the exact solutions were performed and presented.

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