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Graph Theoretical Properties of Degree Six 3-Modified Chordal Ring Networks

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Abstract: Chordal rings are important models for the development of parallel and distributed interconnection networks. Hence, much research on their variations and properties had been published over the years. In this study, a new type of chordal ring network is presented, the degree six 3-modified chordal ring (CHR6m_3) which is able to accommodate both odd and even numbers of nodes so long as the network size is divisible by 3. The aim of the research was to develop relations for the theoretical diameter and average path length for CHR6m_3 as well as to present the main properties of CHR6m_3. The relations were developed based on data from the tree visualisation of CHR6m_3 while the main properties encompass Hamiltonicity, asymmetry and node colouring. Results for theoretical diameter and average path lengths for their corresponding network sizes were based on the formulae generated. Several theorems regarding the main properties were constructed and proven. Hamiltonian circuits could be constructed on the basis of certain types of chords if certain conditions were met. CHR6m_3 was proven to be asymmetric. A proper node colouring was proposed along with the conditions for its existence. A small chromatic number is important in minimising network process completion times.

Key words: Chordal rings, diameter, average path length, hamiltonicity, asymmetry

INTRODUCTION

With the latest advancements in distributive and parallel computing, optimised network planning has become more and more important in assuring good performance parameters. High connectivity results in low latencies when sending and receiving messages among processors. Aside from lowering communication delays, another important factor in determining the efficiency of a network is its fault tolerance or robustness. Thus, it is important to choose a suitable seed topology in the early stages of network planning since, it is not cost efficient or less practical to make changes later on (Bujnowski *et al*., 2011).

One good seed topology that has gained much attention from researches since, its founding by Arden and Lee (1981) is the chordal ring. Based on a ring topology, its connectivity and performance were enhanced by the addition of jump-links called chords, hence providing more paths through which a message can reach its destination processor node. It can be modelled after a graph Gwhere its vertex set V represents the processor nodes in a network and its edge set E represents the network links (Dubalski *et al*., 2012). Each 'edge' between two adjacent 'vertices' in the network plan will then be replaced by two links in opposing directions as stated by Narayanan and Opatrny (1999). The topology can be generalised as an undirected circulant graph denoted by $G(N{:}S{\times}h_1,..., N/2)$ where "s" denotes a ring link and $h_1<...<N/2$ (Farah and Othman, 2014).

The major advantages of the chordal ring were its symmetry which reduced the complexity of routing (Bujnowski *et al*., 2010), its good extensibility (Dubalski *et al*., 2012) and also that it was less susceptible to node or link failure. Aside from multiprocessor interconnection, the chordal ring interconnection topology is also applicable in optical networks such as in Time-Division Multiplexing (TDM) and Wavelength-Division Multiplexing (WDM) (Ledzinski *et al*., 2014).

Previously proposed chordal ring topologies after the first degree three chordal ring (Arden and Lee, 1981) include the degree four traditional chordal ring (Browne and Hodgson, 1990); the degree six traditional chordal ring; the degree five traditional chordal ring (Bujnowski *et al*., 2011) as well as modified chordal rings such as the degree six modified chordal rings (Shah *et al*., 2010) and (Bujnowski *et al*., 2010); the degree five modified chordal ring (Dubalski *et al*., 2012) and the degree four modified chordal ring (Ledzinski *et al*., 2014). Various researchers have been

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investigating the properties of the chordal ring topologies as well such as the symmetric properties of degree three chordal rings (Barriere, 2003) and their optimisation (Morillo *et al*., 1987). The hamiltonicity of degree four traditional chordal rings, more specifically, their 4-ordered property have also been recent focus of research (Kao and Pan, 2015). The performance of chordal rings were compared to that of the 2d torus (Hackett *et al*., 2013) and the former were found to be more robust and performed better under certain circumstances.

MATERIALS AND METHODS

Definition and performance: The degree six 3-modified chordal ring is defined as follows:

Definition 1: The degree six 3-modified chordal ring, $CHR6m_3$ is an undirected circulant graph denoted as $CHR6m_3$ (N, s, h_1, h_2, h_3, h_4). The number of nodes, N must be divisible by 3 and every 3 nodes are grouped into a class. The s is a ring edge. All nodes are connected by the chords $+h_1$ (mod N) and $-h_1$ (mod N).

The first node in the class, N_{3i-3} (e.g., nodes 0, 3, 6,...) Has additional chords $+h_2$ (mod N) which connects it to the third node in one of the successive classes and $+h_3$ (mod N) which connects it to the second node in one of the successive classes.

The second node in the class, N_{3i-2} (e.g., nodes 1, 4, 7,...) Has additional chords $-h_3$ (mod N) which connects it to the first node in one of the perceding classes and $+h_4$ (mod N) which connects it to the third node in one of the successive classes. The third node in the class, N_{3i-1} (e.g., nodes 2, 5, 8,...) has additional chords $-h_2$ (mod N) which connects it to the first node in one of the preceding classes and $-h_4$ (mod N) which connects it to the second node in one of the preceding classes. The i = 1, 2, 3,...N-1. Further conditions are that $h_1 \neq h_2 \neq h_3 \neq h_4$ and h_1, h_2, h_3, h_4<N/2. An example of $CHR6m_3$ is shown in Fig. 1.

A tree visualisation was constructed based on Definition 1. This enabled the determination of the maximum numbers of nodes in its layers and hence, the relations for theoretical diameter and average path lengths. The theoretical diameter of $CHR6m_3$ for N_{do}>15 is given by:

$$D(G)=\frac{1}{10}\left(10\beta+9-5\sqrt{\frac{533}{25}-4\beta^2+\frac{3084}{125\beta}}\right)$$

With:

$$\alpha^3 = 250903-57564\times N_{do}+\frac{1}{2}\sqrt{(501806-115128.N_{do})^2+(68288+2340.N_{do})^3}$$

$$\beta=\sqrt{\frac{533}{300}+\frac{1}{60}\left(\alpha-\frac{360\times N_{do}+1067}{\alpha}\right)}$$

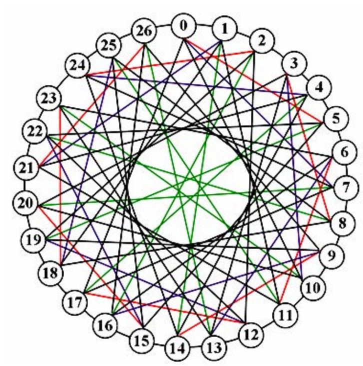

Fig. 1: The $CHR6m_3$ (27, 1, 10, 5, 7, 13)

Table 1: Theoretical diameters and average path lengths for different numbers of nodes

N_{do}	$D(G)_o$	d_{avgo}
18	2	1.589
21	2	1.631
24	2	1.674
27	2	1.716
30	2	1.759
33	2	1.802
36	2	1.844
39	2	1.888
42	2	1.930
45	2	1.971
48	2	2.014
51	2	2.060
54	2	2.104
57	2	2.148
60	2	2.192
63	2	2.237
66	3	2.282
69	3	2.382
72	3	2.374
75	3	2.422
78	3	2.470
81	3	2.520
84	3	2.572
87	3	2.626
90	3	2.682
93	3	2.742
96	3	2.808
99	3	2.881
102	3	2.967

where N_{do} represents the number of nodes in the optimal graph. The average optimal path length of $CHR6m_3$ for D (G)>1 is given by:

$$d_{avo}=\frac{4[D(G)]^5+16[D(G)]^4-12[D(G)]^3+2[D(G)]^2+26[D(G)]}{5[D(G)]^4+18[D(G)]^3-29[D(G)]^2+42[D(G)]}\approx\frac{4}{25}(5[D(G)]+2)$$

Table 1 shows some theoretical results obtained using these relations.

RESULTS AND DISCUSSION

Connectivity and properties: A good measure of connectivity within a graph is the existence of a hamiltonian circuit. A hamiltonian circuit in $CHR6m_3$ visits every node exactly once and returns to its initial node. This means that all links in a particular hamiltonian circuit isolated from $CHR6m_3$ are disjoint.

Theorem: All chord combinations for any $CHR6m_3$ subject to the definition have at least one hamiltonian circuit.

Proof: All the nodes and ring links of $CHR6m_3$ form a regular graph of degree two whereby all nodes are connected to one another.

Theorem: Chord h_1 generates a Hamiltonian circuit if and only if gcd $(N, h_1) = 1$ (h_1 primes N).

Proof: If h_1 does not prime N, gcd (N, h_1) = lcm $(N, h_1) \neq 1$. Or hcf $(N, h_1) = h_1$. In both cases, chord h_1 forms a sub-circuit within $CHR6m_3$ that visits only nodes divisible by [lcm (N, h_1)] mod N. Since, a hamiltonian circuit cannot have a sub-circuit within itself, chord h_1 will not generate a hamiltonian circuit in all cases where gcd $(N, h_1) \neq$ (h_1 does not prime N).

Theorem: Chords h_2, h_3 and h_4 alone will form a hamiltonian circuit if there is no sub-circuit, C_n where $n<N$ generated by some combination of chords h_2, h_3 and h_4.

Proof: A hamiltonian circuit cannot have a sub-circuit within itself.

Theorem: The chromatic number of $CHR6m_3$ is 3 if an only if the length of chord h_1 is not divisible by 3.

Proof: Based on the definition of $CHR6m_3$, its corresponding degree four 3-modified chordal ring formed by s, h_2, h_3 and h_4 for the same number of nodes will always have a 3-colouring where each different node in the class is assigned its own colour, i.e., the first node in the class, N_{3i-3} is labelled as red, the second node in the class, N_{3i-2} is labelled as yellow and the third node in the class, N_{3i-1} is labelled as green. Whether or not $CHR6m_3$ has a 3-colouring depends on chord h_1. In this case, chord h_1 must not be divisible by 3 to preserve the aforementioned proper labelling.

The asymmetrical property of a modified chordal ring refers to the case where the modified chordal ring that is neither node-symmetric nor link-symmetric but a small symmetry can be seen when nodes are divided into classes, in contrast to symmetry where the entire graph looks the same when viewed from any node (Farah *et al*., 2010). Node-symmetry means that every pair of source and destination nodes is similar.

Table 2: Source nodes and their corresponding destination nodes

Source nodes	Destination nodes 1	2	3	4
N_{3i-3}	$N_{3i-3\pm s}$	$N_{3i-3\pm h1}$	$N_{3i-3+h2(mod\ N)}$	$N_{3i-3+h3(mod\ N)}$
N_{3i-2}	$N_{3i-2\pm s}$	$N_{3i-2\pm h1}$	$N_{3i-2-h3(mod\ N)}$	$N_{3i-2+h4(mod\ N)}$
N_{3i-1}	$N_{3i-1\pm s}$	$N_{3i-1\pm h1}$	$N_{3i-1h2(mod\ N)}$	$N_{3i-1-h4(mod\ N)}$

Destination nodes with nodes with respect to source nodes respect to which are not similar in each source nodes class which are similar in each class

Table 3: Source nodes and the links leading to their corresponding destination nodes

Source nodes	Links corresponding destination nodes (ring and chords) 1	2	3	4
N_{3i-3}	$\pm s$	$\pm h_1$	$+h_2$	$+h_3$
N_{3i-2}	$\pm s$	$\pm h_1$	$-h_3$	$+h_4$
N_{3i-1}	$\pm s$	$\pm h_1$	$-h_2$	$-h_4$

Theorem: The $CHR6m_3$ is not node-symmetric.

Proof: Table 2 shows the generalised source and destination nodes for $CHR6m_3$. It can be observed that not every pair of source and destination nodes is similar. Link-symmetry, on the other hand, means every pair of links between source and destination nodes are similar.

Theorem: The $CHR6m_3$ is not link-symmetric.

Proof: Table 3 shows the links between source and destination nodes. The s, h_1, h_2, h_3 and h_4 are elements of the link set H.

It can be observed that for all source nodes, there exist an automorphism π of $CHR6m_3$ such that $\pi(\pm s) = \pm s$ and $\pi(\pm h_1) = \pm h_1$. Even though, there exist an automorphism π of $CHR6m_3$ such that $\pi(+h_2) = -h_2$ for source nodes N_{3i-3} and N_{3i-1}, $\pi(+h_3) \neq -h_4$.

Even though, there exist an automorphism π of $CHR6m_3$ such that $\pi(+h_3) = -h_3$ for source nodes N_{3i-3} and N_{3i-2}, $\pi(+h_2) \neq +h_4$. Even though, there exist an automorphism π of $CHR6m_3$ such that $\pi(+h_4) = -h_4$ for source nodes N_{3i-3} and N_{3i-2}, $\pi(-h_2) \neq -h_3$. Hence, not every edge linking source nodes of every class to their destination nodes in $CHR6m_3$ is similar. The $CHR6m_3$ is asymmetric as it is not node-symmetric and not link-symmetric.

CONCLUSION

In this study, we have presented a new approach in degree six modified chordal rings which enabled them to be applied to networks of both odd and even sizes, so long as the number of nodes is divisible by 3. Hamiltonicity is a very important measure of connectivity and robustness within a network. The presence of multiple hamiltonian circuits ensures that the network remains connected in the event of link failures. The properties of asymmetry and the suggested proper colouring of $CHR6m_3$ will aid the development of a routing algorithm, processor task scheduling and designation of memory units in parallel networks. Further research should explore routing in $CHR6m_3$.

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