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A Weak Convergence Theorem for Variational Inequalities and Fixed Point Problems in a Real Hilbert Space

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Abstract: In this study, using the projection technique, we introduce a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem for inverse strongly monotone mapping in a real Hilbert space.

Key words: Weak convergence theorem, real hilbert space, variational inequality theory, element, India

INTRODUCTION

Variational inequalities were introduced by Stampacchia (1964). In recent years, variational inequality theory has been extended and generalized in several directions. One of the most important problem in this theory is the development of an efficient iterative scheme for solving variational inequalities. Also we have the problem of finding the fixed points of nonexpansive mappings which is the subject of current interest in functional analysis. Therefore it is natural to consider a unified approach to these two different problems. Let C be a closed convex subset of a real Hilbert space H and P_C be the metric projection of H onto C. A mapping A of C into H is called monotone if:

$$\langle Au-Av, u-v \rangle \geq 0, \text{ for all } u, v \in C$$

A mapping A of C into H is called α-inverse-strongly-monotone (Browder and Petryshyn, 1967; Iiduka and Takahashi, 2005; Liu and Nashed, 1998) if there exists a positive real number α such that:

$$\langle Au-Av, u-v\rangle \geq \alpha\|Au - Av\|^2, \ \forall u, v \in C$$

It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. The variational inequality problem is to find u ∈ C such that (Lions and Stampacchia, 1967; Browder, 1965; Bruck, 1977; Takahashi, 1978):

$$\langle Au,v-u\rangle \geq 0, \text{ for all } u \in C$$

The set of solutions of variational inequality problem is denoted by VI (C, A). A mapping T of C into itself is called nonexpansive (Goebel and Kirk, 1990; Takahashi, 2000) if:

$$\|Tu - Tv\| \leq \|u - v\|, \ \forall u, v \in C$$

We denote by F(T) the set of fixed points of T. For finding an element of F(S)∩VI(C, A), Takahashi and Toyada (2003) gave the following result.

Theorem 1.1: Let C be a closed convex subset of a real Hilbert space H (Takahashi and Toyoda, 2003). Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that F (S) ∩ VI(C,A) ≠ Φ. Let $\{x_n\}$ be a sequence generated by:

$$x_0 = x \in C$$

$$x_n+1 = \alpha_n x_n + (1-\alpha n)SP_C(x_n - \lambda_n Ax_n) \quad (1)$$

for every n = 0, 1, 2, ... where $\{\lambda_n\} \subset [a, b]$ for some a, b∈ (0, 2α) and $\{\alpha_n\} \subset [c,d]$ for some c, d ∈(0, 1). Then, the sequence $\{x_n\}$ converges weakly to some point z∈F(S)∩ VI(C, A), where:

$$z = \lim_{n\to\infty} P_{F(S)\cap VI(C, A)} x_n$$

After that for finding a common element of F(S)∩ VI(C, A), Nadezhkina and Takahashi (2006) gave another result. They obtained the following weak convergence theorem.

Theorem 1.2: Let C be a closed convex subset of a real Hilbert space H (Nadezhkina and Takahashi, 2006). Let A be a monotone and k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that F(S)∩VI(C, A) ≠ Φ. Let $\{x_n\}$, $\{y_n\}$ be sequences generated by:

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$$x_0 = x \in C$$

$$y_n = P_c(x_n - \lambda_n A x_n)$$

$$x_n+1 = \alpha_n x_n + (1 - \alpha_n) S P_c(x_n - \lambda_n A y_n),\ \forall\ n \geq 0 \quad (2)$$

Where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ generated by (1.2) converge weakly to some $z \quad F(S) \cap VI(C, A)$. Also, Noor (2000, 2007) suggested the following three step iterative scheme. For given $x_0 \in C$, the approximate solution x_{n+1} is given by:

$$y_n = P_c(x_n - \lambda A x_n)$$

$$z_n = P_c(y_n - \lambda_n A y_n)$$

$$x_n+1 = P_c(z_n - \lambda A z_n),\ \forall\ n \geq 0,\ \lambda \in (0, \frac{1}{k})$$

MATERIALS AND METHODS

Preliminaries: Let H be a real Hilbert space with inner product <., .> and norm ‖.‖. Let C be a closed convex subset of H. We shall write xn→x to show that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We know that for any $x \in H$, there exists a unique nearest point u in C, such that:

$$\|u - P_C x\| = \inf \{\|u - y\| : y \in C\} \quad (1)$$

P_C is called the metric projection of H onto C. The metric projection P_C of H onto C satisfies:

$$< x - y, P_C x - P_C y > = \|P_C x - P_C y\|^2, \text{ for every } x, y \in H \quad (2)$$

P_C is characterized by the property:

$$P_C x \in C$$

Also, P_C satisfies the following properties:

$$< x - P_C x, y - P_C x > \leq 0, \text{ for all } x \in H, y \in C \quad (3)$$

$$\|x \in y\|^2 = \|x - P_C x\|^2 + |y - P_C x|^2, \text{ for all } x \in H, y \in C \quad (4)$$

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see that:

$$u \in \Omega \Leftrightarrow u = P_C (u - \lambda A u), \text{ for any } \lambda > 0 \quad (5)$$

Where Ω is the set of solutions of variational inequality problem. It is known that H satisfies (1967) the Opial condition (Takahashi, 2000) that is, for any sequence $\{x_n\}$ with xn→x, the inequality:

$$\lim_{n\to\infty} \inf \in \|xn - x\| < \lim_{n\to\infty} \inf \in \|xn - y\| \quad (6)$$

holds for every $y \in H$ with $y \neq x$. We also know that, if $\{x_n\}$ is sequence of H with $x_n \to x$ and $\|x_n\| \quad \|x\|$, then this gives that $x_n \to x$. A set valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, f Tx and g Ty im ply < x-y, f-g >≥ 0. A monotone mapping $T: H \to 2^H$ is maximal if its graph G(T) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, < x-y, f-g >≥ 0 for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let A: C → H be a monotone, k-Lipschitz continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, that is:

$$N_C v = \{w \in H : < v - u, w > = 0, u \in C\}$$

Define:

$$Tv = \begin{cases} Av + N_c v, & \text{if } v \in C \\ \phi, & \text{if } v \notin C \end{cases}$$

Then, T is maximal monotone (Rockafellar, 1970, 1976) and $0 \in Tv$ if and only if $v \in VI(C, A)$. If A is an α-inverse-strongly-monotone mapping of C into H, then it is clear that A is 1/α-Lipschitz continuous. We also know that for all $x, y \in C$ and $\lambda > 0$:

$$\begin{aligned} &\|(I - \lambda A)x - (I - \lambda A)y\|^2 = \\ &\|(x - y) \in \lambda(Ax - Ay)\|^2 = \\ &\|x \in y\|^2 - 2\lambda < x \in y, \\ &Ax \in Ay > + \lambda^2 \|Ax \in Ay\|^2 = \\ &\|x \in y\|^2 + \lambda(\lambda - 2\alpha) \|Ax \in Ay\|^2 \end{aligned} \quad (7)$$

So, if $\lambda \leq \alpha$, then I-λ A is a nonexpansive mapping of C into H. Takahashi and Toyada, gave the following result for the existence of solutions of the variational inequality problem for α-inverse-strongly-monotone mappings.

Proposition: Let C be a bounded closed convex subset of a real Hilbert space H and let A be an α-inverse-strongly-monotone mapping of C into H (Takahashi and Toyoda, 2003). Then, VI (C, A) is non-empty. Now we give some lemmas which will be useful to prove our main result. The first lemma was given by Schu (1991).

Lemma 2.1: Let H be a real Hilbert space and let $\{\alpha_n\}$ be a sequence of real numbers such that $0\ a \leq \alpha_n \leq b<1$ for all n = 0, 1, 2… and let $\{v_n\}$ and $\{w_n\}$ be sequences of H such that (Schu, 1991):

$$\lim_{n\to\infty} \sup\in \|v_n\| = c, \lim_{n\to\infty} \sup\in \ |w_n| = c$$

And:

$$\lim_? (n \to \infty) \|\alpha_n v_n + (1-\alpha_n) w_n\| = c,$$
for some c > 0

Then:

$$\lim_? (n \to \infty) \|vn - wn\| = 0$$

Next lemma was given by Takahashi and Toyada (2003).

Lemma 2.2: Let H be a real Hilbert space and let D be a non empty closed convex subset of H (Takahashi and Toyada, 2003). Let $\{x_n\}$ be a sequence in H. Suppose that, for all u∈D:

$$\|x_{n+1} - u\| = \|x_n - u\|$$

for every n = 0, 1, 2, … Then, the sequence $\{P_D x_n\}$ converges strongly to some z∈D.

Lemma 2.3: Let H be a real Hilbert space, C be a nonempty closed convex subset of H and T: C→C be a nonexpansive mapping (Goebel and Kirk, 1990). Then, the mapping I-T is demiclosed on C, where I is the identity mapping; that is, $x_n \to x$ in C and (I-T) $x_n \to y$ imply that x∈C and (I-T) x = y.

RESULTS AND DISCUSSION

Weak convergence theorem: Now, we give a new iterative scheme for nonexpansive and inverse strongly monotone mappings and prove a weak convergence theorem for this scheme.

Theorem: Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that F(S) ∩ VI(C, A)≠φ. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by:

$$x_0 = x \in C,\ z_n = P_C(x_n - \tau_n A x_n),$$
$$y_n = P_C(z_n - \mu_n A z_n),$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(y_n - \lambda_n A y_n),\ n = 0$$

where, $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\tau\}_n$ satisfy the following conditions:

- $\{\alpha_n\}$ is a sequence in (0, 1)
- $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\tau_n\}$ are three sequences in [a, b] for some a, b ϵ (0, 2α)

Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ will converge weakly to the same point z ε F(S) ∩ VI(C, A), where:

$$z = \lim_{n\to\infty} P_{F(S) \cap VI(C, A)}\ x_n$$

Proof: For all x, y ϵ C and λ_n ϵ (0, 2α), we have:

$$\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 = \|(x-y) \in \lambda_n (Ax - Ay)\|^2$$
$$= \|x \in y\|^2 - 2\lambda_n < x \in y,$$
$$Ax \in Ay > + \lambda_n\ 2|\ Ax \in Ay|^2 =$$
$$\|x \in y\|^2 + \lambda_n\ (\lambda_n - 2\alpha)\|Ax \in Ay\|^2 \quad (1)$$

The I-λ_nA is a nonexpansive mapping of C into H. Let uϵF(S)∩VI(C, A), then we have $u = P_C(u-\lambda Au)$, for all λ>0. Note that:

$$\|z_n \in u\| \leq \|P_C(x_n - \tau_n A x_n) \in PC(u - \tau_n Au)\|$$
$$\leq \|x_n \in u\|$$

And:

$$\|y_n - u\| \leq \|P_C(z_n - \mu_n A z_n) - P_C(u - \mu_n Au)\|$$
$$\leq \|z_n - u\|$$
$$\leq x_n - u .$$

Also, let $w_n = P_C(y_n - \lambda_n A y_n)$. Then:

$$\|w - u\| \leq \|P_C(y_n - \lambda_n A y_n) - P_C(u - \lambda_n Au)\|$$
$$\leq \|y_n - u\|$$
$$\leq \|x_n - u\| .$$

Now:

$$\|x_{n+1} - u\| = \|\alpha_n (x_n - u) + (1-\alpha_n)(Sw_n - u)\|$$
$$\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\|w_n - u\|$$
$$\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\|x_n - u\|$$
$$= \|x_n - u\|$$

So, $|x_{n+1} - u| = |x_n - u|$. Therefore, there exists $c = \lim_{n\to 8} |x_n - u|$ and the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ are bounded. Now:

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n(x_n - u) + (1-\alpha_n)(Sw_n - u)\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)|y_n - u|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Au\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|x_n - u\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Ay_n - Au\|^2\\ &\le \|x_n - u\|^2 + (1-\alpha_n)b(b-2\alpha)\|Ay_nAu\|^2\\ \Rightarrow\quad & -(1-\alpha_n)b(b-2\alpha)\|Ay_n - Au\|^2\\ &\le \|x_n - u\|^2 - \|x_{n+1} - u\|^2\end{aligned}$$

Since:

$$\lim_{n\to\infty}\|x_n - u\| = \lim_{n\to\infty}\|x_{n+1} - u\|, \text{ so } \lim_{n\to\infty}\|Ay_n - Au\| = 0 \tag{2}$$

On the other hand, we have:

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n(x_n - u) + (1-\alpha_n)(Sw_n - u)\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|w_n - u\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)(\|z_n - u\|^2 + \mu_n(\mu_n - 2\alpha)\|Az_n - Au\|^2)\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|x_n - u\|^2 + (1-\alpha_n)\mu_n(\mu_n - 2\alpha)\|Az_n - Au\|^2\\ &\le \|x_n - u\|^2 + (1-\alpha_n)b(b-2\alpha)\|Az_n - Au\|^2\\ \Rightarrow\quad & -(1-\alpha_n)b(b-2\alpha)\|Az_n - Au\|^2\\ &\le \|x_n - u\|^2 - \|x_{n+1} - u\|^2\end{aligned}$$

Since:

$$\lim_{n\to\infty}\|x_n - u\| = \lim_{n\to\infty}\|x_{n+1} - u\|, \text{ so } \lim_{n\to\infty}\|Az_n - Au\| = 0 \tag{3}$$

In a similar way, we can obtain that:

$$\operatorname{Lim}_{n\to\infty}\|Ax_n - Au\| = 0 \tag{4}$$

$$\begin{aligned}\|w_n - u\|^2 &= \|P_C(y_n - \lambda_n Ay_n) - P_C(u - \lambda_n Au)\|^2\\ &\le \langle y_n - \lambda_n Ay_n - (u - \lambda_n Au), w_n - u\rangle\\ &= \frac{1}{2}\{\|y_n - \lambda_n Ayn - (u - \lambda_n Au)\|^2 + \|w_n - u\|^2 - |y_n - \lambda_n Ay_n - (u - \lambda_n Au) - (w_n - u)|^2\}\\ &\le \frac{1}{2}\{\|y_n - u\|^2 + \|w_n - u\|^2 - \|y_n - w_n\| - \lambda_n(Ay_n - Au)|^2\}\\ &\le \frac{1}{2}\{\|y_n - u\|^2 + \|w_n - u\|^2 - \|y_n - w_n\|^2 + 2\lambda_n\langle y_n - w_n, Ay_n - Au\rangle - \lambda_n^2\|Ay_n - Au\|^2\}\end{aligned}$$

Therefore, we obtain:

$$\|w_n - u\|^2 \le \|y_n - u\|^2 - \|y_n - w_n\|^2 + 2\lambda_n\langle y_n - w_n, Ay_n - Au\rangle - \lambda_n^2\|Ay_n - Au\|^2$$

On the other hand, we have:

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n(x_n - u) + (1-\alpha_n)(Sw_n - u)\|^2\\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|w_n - u\|^2\end{aligned}$$

$$\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\{\|y_n - u\|^2 - \|y_n - w_n\|^2 + 2\lambda_n < y_n - w_n , Ay_n - Au> - \lambda_n{}^2 \|Ay_n - Au\|^2$$
$$\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|x_n - u\|^2 - \|y_n - w_n\|^2 + 2\lambda_n < y_n - w_n , Ay_n - Au> - \lambda_n{}^2 \|Ay_n - Au\|^2$$
$$= \|x_n - u\|^2 - \|y_n - w_n\|^2 + 2\lambda_n < y_n - w_n , Ay_n - Au> - \lambda_n{}^2 \|Ay_n - Au\|^2$$

By Eq. 2 and since $\lim_{n\to\infty}\|x_n - u\| = \lim_{n\to\infty}\|x_{n+1} - u\|$, so:

$$\lim_{n\to\infty} \|y_n - w_n\| = 0 \tag{5}$$

In the same way, we obtain:

$$\|y_n - u\|^2 = \|P_C(z_n - \mu_n Az_n) - P_C(u - \mu_n Au)\|^2$$
$$\leq < z_n - \mu_n Az_n - (u - \mu_n Au), y_n - u>$$
$$= \frac{1}{2}\{\| z_n - \mu_n Az_n - (u - \mu_n Au)\|^2 + \|y_n - u\|^2 - \|z_n - \mu_n Az_n - (u - \mu_n Au) - (y_n - u)\|^2\}$$
$$\leq \frac{1}{2} \{\|z_n - u\|^2 + \|y_n - u\|^2 - \|(z_n - y_n) - \mu_n (Az_n - Au)\|^2\}$$
$$\leq \frac{1}{2} \{\|z_n - u\|^2 + \|y_n - u\|^2 - \|z_n - y_n\|^2 + 2\mu_n < z_n - y_n, Az_n - Au> - \mu_n{}^2 \|Az_n - Au\|^2\}$$

from which it follows that:

$$\|y_n - u\|^2 \leq \|z_n - u\|^2 - \|z_n - y_n\|^2 + 2\mu_n < z_n - y_n, Az_n - Au> - \mu_n{}^2 \|Az_n - Au\|^2$$

Now:

$$\|x_{n+1} - u\|^2 = \|\alpha_n(x_n - u) + (1-\alpha_n)(Sw_n - u)\|^2$$
$$\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|w_n - u\|^2$$
$$\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2$$
$$\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\{\|z_n - u\|^2 - \|z_n - y_n\|^2 + 2\mu_n < z_n - y_n, Az_n - Au> - \mu_n{}^2 \|Az_n - Au\|^2 \tag{6}$$
$$\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|x_n - u\|^2 - \|z_n - y_n\|^2 + 2\mu_n < z_n - y_n, Az_n - Au> - \mu_n{}^2 \|Az_n - Au\|^2$$
$$= \|x_n - u\|^2 - \|z_n - y_n\|^2 + 2\mu_n < z_n - y_n, Az_n - Au> - \mu_n{}^2| \|Az_n - Au\|^2$$

By Eq. 3 and since $\lim_{n\to\infty}\|x_n - u\| = \lim_{n\to\infty}\|x_{n+1} - u\|$, so:

$$\lim_{n\to\infty} \|z_n - y_n\|^2 = 0 \tag{6}$$

On the other hand, we have:

$$\|z_n - u\|^2 = \|P_C(x_n - \tau_n Ax_n) - P_C(u - \tau_n Au)\|^2$$
$$\leq < x_n - \tau_n Ax_n - (u - \tau_n Au), z_n - u >$$
$$= \frac{1}{2} \{\| x_n - \tau_n Ax_n - (u - \tau_n Au)\|^2 + \|z_n - u\|^2 - \|x_n - \tau_n Ax_n - (u - \tau_n Au) - (z_n - u)\|^2\}$$
$$\leq \{\|x_n - u\|^2 + \|z_n - u\|^2 - \|x_n - z_n\|^2 + 2\tau_n < x_n - z_n, Ax_n - Au> - \tau_n{}^2 \|Ax_n Au\|^2\}$$

which implies that:

$$\|z_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - z_n\|^2 + 2\tau_n < x_n - z_n , Ax_n - Au> - \tau_n{}^2 \|Ax_n - Au\|^2$$

Hence, we have:

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n (x_n - u) + (1 - \alpha_n)(Sw_n - u)\|^2 \\ &\le \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|w_n - u\|^2 \\ &\le \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\ &\le \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|z_n - u\|^2 \\ &\le \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 - \|x_n - z_n\|^2 + 2\tau_n < x_n - z_n, Ax_n - Au> - \tau_n{}^2 \|Ax_n - Au\|^2\} \\ &\le \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 - \|x_n - z_n\|^2 + 2\tau_n < x_n - z_n, Ax_n - Au> - \tau_n{}^2 \|Ax_n - Au\|^2\} \\ &= \|x_n - u\|^2 - \|x_n - z_n\|^2 + 2\tau_n < x_n - z_n , Ax_n - Au> - \tau_n{}^2 \|Ax_n - Au\|^2\}\end{aligned}$$

By Eq. 4 and since $\lim_{n\to\infty}\|x_n - u\| = \lim_{n\to\infty}\|x_{n+1} - u\|$, so:

$$\lim_{n\to\infty}\|x_n - z_n\| = 0 \qquad (7)$$

Also, by Eq. 6 and 7, we have:

$$\|x_n - y_n\| \le \|x_n - z_n\| + \|z_n - y_n\| \to 0 \text{ as } n \to \infty \qquad (8)$$

Now by Eq. 5 and 7, we have:

$$\|x_n - w_n\| = \|x_n - y_n\| + \|y_n - w_n\| \to 0 \text{asn} \to \infty. \qquad (9)$$

As $\{x_n\}$ is bounded, so we have a subsequence $\{x_{ni}\}$ of $\{x_n\}$ that converges weakly to some z. We shall show that:

$$z \in F(S) \cap VI(C, A)$$

Firstly, we shall show that $z \in VI(C, A)$. Since, $x_n - y_n \to 0$, $x_n - z_n \to 0$, $x_n - w_n \to 0$, so we have $z_{ni} \to z$, $y_{ni} \to z$, $x_{ni} \to z$. Let:

$$Tv = \begin{cases} Av + N_c & \text{if } v \in c \\ \phi & \text{if } v \notin c \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(T)$. Since, $w - Av \in N_C v$ and $w_n \in C$, so we have:

$$<v - w_n , w - Av > \ge 0$$

On the other hand, from:

$$w_n = P_C(y_n - \lambda_n Ay_n)$$

We have:

$$<v - w_n , w_n - (y_n - \lambda_n Ay_n)> \ge 0$$

And hence:

$$<v - w_n, (w_n - y_n)/\lambda_n + Ay_n> \ge 0$$

Therefore:

$$\begin{aligned}&< v - w_{n_i}, w> \\ &\ge < v - w_{n_i}, Av> \\ &\ge < v - w_{n_i}, Av > - < v - w_{n_i}, (w_{n_i} - y_{n_i})/\lambda_{n_i} + A> \\ &= < v - w_{n_i}, Av - Ay_{n_i} - (w_{n_i} - y_{n_i})/ \lambda_{n_i}> \\ &= < v - w_{n_i}, Av - Aw_{n_i} > + < v - w_{n_i}, Aw_{n_i} - Ay_{n_i} > - \\ &\quad < v - w_{n_i}, (w_{n_i} - y_{n_i})/\lambda_{n_i}> \\ &\ge < v - w_{n_i}, Aw_{n_i} - Ax_{n_i} > - < v - w_{n_i}, (w_{n_i} - y_{n_i})/\lambda_{n_i}>\end{aligned}$$

Hence, we obtain:

$$< v - z, w > \ge 0, \text{ as } i \to \infty$$

Since, T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Now, we shall show that $z \in F(S)$. Let $u \in F(S) \cap VI(C, A)$. Since:

$$\|Sy_n - u\| = \|y_n - u\| = \|x_n - u\|$$

So, we have:

$$\lim_{n\to\infty} \sup \|Sy_n - u\| \le c$$

Where:

$$c = \lim_{n\to\infty} \|x_n - u\|$$

Further, we have:

$$\lim_{n\to\infty}\|\alpha_n(x_n - u) + (1 - \alpha_n)(Sy_n - u)\|$$
$$= \lim_{n\to\infty}\|x_{n+1} - u\|$$
$$= c$$

By lemma 2.1 ,we have:

$$\lim_{n\to\infty}\|Sy_n - x_n\| = 0$$

We also have:

$$\|Sx_n - x_n\| \le \|Sx_n - Sy_n\| + \|Sy_n - x_n\|$$
$$\le \|x_n - y_n\| + \|Sy_n - x_n\|$$

Hence, we have:

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0$$

Since, $x_{ni} \to z$ and $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$, so by demiclosedness of I-S, we have $z \in F(S)$. Let $\{x_{nj}\}$ be another subsequence of $\{x_n\}$, such that $\{x_{nj}\} \to z'$. Then, $z' \in F(S) \cap VI(C,A)$. Let us show that $z = z'$. Assume that $z \ne z'$. From the Opial condition, we have:

$$\lim_{n\to\infty}\|x_n - z\| = \liminf_{n\to\infty}\|x_{n_i} - z\| < \liminf_{n\to\infty}\|x_{n_i} - z'\|$$
$$= \lim_{n\to\infty}\|x_n - z'\| = \liminf_{n\to\infty}\|x_{n_j} - z'\|$$
$$< \liminf_{n\to\infty}\|x_{n_j} - z\| = \lim_{n\to\infty}\|x_n - z\|$$

This is a contradiction. Thus we have $z = z'$. This implies:

$$x_n \to z \in F(S) \cap VI(C, A)$$

Now, put:

$$u_n = P_{F(S) \cap VI(C, A)}x_n$$

We show that:

$$z = \lim_{n\to\infty} u_n$$

From:

$$u_n = P_{F(S)\cap VI(C, A)}x_n \text{ and } z \in F(S) \cap VI(C, A)$$

We have:

$$<z - u_n, u_n - x_n> \ge 0$$

By lemma 2.2, $\{u_n\}$ converges strongly to some $z_0 \in F(S) \cap VI(C, A)$. Then, we have:

$$<z - z_0, z_0 - z> \ge 0$$

And hence $z = z_0$.

Applications: Using theorem 3.1, we shall prove two theorems as by Iiduka and Takahashi (2005), Moudafi (2000), Xu (2004) and Chen *et al*. (2007). A mapping T: C> C is called strictly pseudocontractive if there exists α with $0 \le \alpha \le 1$ such that:

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \alpha\|(I - T)x - (I - T)y\|^2$$

For every x, $y \in C$. if $\alpha = 0$, then T is nonexpansive. Put A = I-T, where T: C>C is a strictly pseudocontractive mapping with α. Then, A is $(1-\alpha)/2$-inverse-strongly-monotone. Actually, by definition of T, we have that for all x, $y \in C$:

$$\|(I - A)x - (I - A)y\|^2 \le \|x - y\|^2 + \alpha\|Ax - Ay\|^2$$

On the other hand, we have:

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2<x - y, Ax - Ay>$$

Hence, we have:

$$<x - y, Ax - Ay> \ge (1 - \alpha)/2\|Ax - Ay\|^2$$

Using theorem 3.1, we first present a weak convergence theorem for a pair of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 4.1: Let C be a closed convex subset of a real Hilbert space H, let S be a nonexpansive mapping of C into itself and let T be a strictly pseudocontractive mapping of C into itself such that $F(S) \cap F(T) \ne \Phi$. Let $\{x_n\}$ be a sequence generated by:

$$x_0 = x \in C$$
$$z_n = (1 - \tau_n)x_n + \tau_n Tx_n$$
$$y_n = (1 - \mu_n)z_n + \mu_n Tz_n$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S((1 - \lambda_n)y_n + \lambda_n Ty_n), n = 0$$

where $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\tau_n\}$ satisfy the following conditions:

- $\{\alpha_n\}$ is a sequence in (0, 1)
- $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\tau_n\}$ are three sequences in [a, b] for some a, $b \in (0, 2\alpha)$

Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ will converge weakly to the same point $z \in F(S) \cap VI(C, A)$, where:

$$z = \lim_{n\to\infty} P_{F(S)} \cap VI(C, A)\, x_n$$

Proof: Put A = I-T. Then, A is inverse strongly monotone. We have that:

$$F(T) = VI(C,A)$$

And:

$$P_C(y_n - \lambda_n A y_n) = (1-\lambda_n)\, y_n + \lambda_n T y_n$$

Thus, the desired result can be obtained from theorem 3.1. Using theorem 3.1, we also have the following result.

Theorem 4.2: Let H be a real Hilbert space. Let A be an α-inverse-strongly-monotone mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $F(S)\ n\ A^{-1}0 \neq \Phi$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in H$ and let:

$$z_n = x_n - \tau_n A x_n$$
$$y_n = z_n - \mu_n A z_n$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S(y_n - \lambda_n A y_n),\ n = 0$$

where, $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\tau_n\}$ satisfy the following conditions:

- $\{\alpha_n\}$ is a sequence in (0, 1)
- $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\tau_n\}$ are three sequences in [a, b] for some a, $b \in (0, 2\alpha)$

Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge weakly to the same point $z \in F(S) \cap A^{-1}0$, where:

$$z = \lim_{n\to\infty} P_{F(S)\cap A^{-1}0}\, x_n$$

Proof: We have $A^{-1}0 = VI(H, A)$ and $P_H = I$. By theorem 3.1, we obtain the desired result.

Remark. Notice that $F(S)\ n\ A^{-1}0 \subset VI(F(S), A)$. Yamada (2001) for the case when A is strongly monotone and Lipschitz continuous mapping of H into itself.

CONCLUSION

In this study, motivated and inspired by above mentioned results, we introduce a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem for inverse strongly monotone mapping in a real Hilbert space. We shall obtain a weak convergence theorem for the three sequences generated by this scheme. Using this result, we shall obtain a weak convergence theorem for a pair of a nonexpansive mapping and a strictly pseudocontractive mapping. Further, we consider the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse strongly monotone mapping.

REFERENCES

Browder, F.E. and W.V. Petryshyn, 1967. Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl., 20: 197-228.

Browder, F.E., 1965. Nonlinear monotone operators and convex sets in Banach spaces. Bull. Am. Math. Soc., 71: 780-785.

Bruck, R.E., 1977. On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space. J. Math. Anal. Appl., 61: 159-164.

Chen, J., L. Zhang and T. Fan, 2007. Viscosity approximation methods for nonexpansive mappings and monotone mappings. J. Math. Anal. Appl., 334: 1450-1461.

Goebel, K. and W.A. Kirk, 1990. Topics in Metric Fixed Point Theory, Vol. 28 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, UK.,.

Iiduka, H. and W. Takahashi, 2005. Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings. Nonlinear Anal. Theory Methods Appl., 61: 341-350.

Lions, J.L. and G. Stampacchia, 1967. Variational inequalities. Commun. Pure Applied Math., 20: 493-519.

Liu, F. and M.Z. Nashed, 1998. Regularization of nonlinear ill-posed variational inequalities and convergence rates. Set Valued Anal., 6: 313-344.

Moudafi, A., 2000. Viscosity approximation methods for fixed-points problems. J. Math. Anal. Appl., 241: 46-55.

Nadezhkina, N. and W. Takahashi, 2006. Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl., 128: 191-201.

Noor, M.A., 2000. New approximation schemes for general variational inequalities. J. Math. Anal. Appl., 251: 217-229.

Noor, M.A., 2007. General variational inequalities and nonexpansive mappings. J. Math. Anal. Appl., 331: 810-822.

Opial, Z., 1967. Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc., 73: 591-597.

Rockafellar, R.T., 1970. On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc., 149: 75-88.

Rockafellar, R.T., 1976. Monotone operators and the proximal point algorithm. SIAM. J. Control Optim., 14: 877-898.

Schu, J., 1991. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc., 43: 153-159.

Stampacchia, G., 1964. Formes bilineaires coercitives sur les ensembles convexes. CR. Acad. Sci. Paris, 258: 964-964.

Takahashi, W. and M. Toyoda, 2003. Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl., 118: 417-428.

Takahashi, W., 2000. Nonlinear Functional Analysis Fixed Point Theory and Applications. Yokohama Publisher, Yokohama, Japan,.

Takahashi, W., H. Iiduka and M. Toyoda, 2004. Approximation of solutions of variational inequalities for monotone mappings. Panamer. Math. J., 14: 49-61.

Takahashi, W.A.T.A.R.U., 1978. Nonlinear complementarity problem and systems of convex inequalities. J. Optim. Theory Appl., 24: 499-506.

Xu, H., 2004. Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl., 298: 279-291.

Yamada, I., 2001. The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings. Inherently Parallel Algorithms Feasibility Optim. Their Appl., 8: 473-504.