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Upper Convected Maxwell Flow Due to a Solid Rod Oscillating with Different Frequencies

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Abstract: The motion of a viscoelastic, incompressible flow of an upper-convected Maxwell fluid, due to the motion of a long, straight, solid, circular cylinder, oscillating both longitudinally and torsionally with different frequencies is examined. The flow is considered to be fully established and so start up effects are ignored. Analytical expressions for the velocity field, the tangential drag and the work done by the drag force have been obtained and the corresponding Newtonian cases deduced. The velocity components and work done are displayed graphically using particular values of the flow parameters. These are compared with Newtonian fluids so as to get some insight into the effects of elasticity.

Key words: Oscillation, upper convected Maxwell, viscoelastic, longitudinal, torsional, different frequencies, Newtonian

INTRODUCTION

There are many fluids which cannot be explained by the classical Newtonian theory, some of these fluids are known as non-Newtonian fluids. Viscoelastic fluids fall into such a class, exhibiting both viscous and elastic characteristics; consequently, they have a time dependent strain. In order to determine their stress and strain interactions, as well as their temporal dependencies, they need to be modelled. One such fluid model is that known as the Upper Convected Maxwell fluid, U.C.M. (Petrie, 1979).

Casarella and Laura (1969) investigated the motion of a viscous fluid due to a circular cylindrical rod immersed in it, which was infinite in length and undergoing longitudinal and torsional oscillations of the same frequency. They were primarily interested in the drag force acting on the rod as this is of practical significance in many ocean engineering problems. Rajagopal (1983) examined a similar problem for a non-Newtonian fluid, that is, a fluid of second grade and like Casarella and Laura (1969) obtained an exact solution for the field equation. Ramkissoon and Majumdar (1990) examined the corresponding internal flow problem for viscous fluids, they obtained an exact solution for the flow field and explicit expressions were given for the shear stresses, the drag experienced by the cylinder and the drag coefficient. Ramkissoon *et al.* (1991) examined the same problem as Casarella and Laura (1969) but for a Polar fluid, here an exact solution was obtained for the velocity field. Rahaman (2004, 2005) examined the same problems as Casarella and Laura (1969) and Ramkissoon and Majumdar (1990), but for an Upper Convected Maxwell fluid, explicit expressions were obtained in each case for the velocity field, shear stresses and drag. Owen and Rahaman (2006) examined the corresponding internal flow problem for an Oldroyd-B fluid, but subjected to torsional and longitudinal oscillations of different frequencies, explicit expressions were obtained for the velocity field, shear stresses and drag.

The main objective of this research is to investigate the motion of a viscoelastic, incompressible flow of the Upper Convected Maxwell fluid, due to a long, straight, solid, circular cylinder, oscillating both longitudinally and torsionally at different frequencies and in the absence of body forces. As in previous research done, the flow is considered to be fully developed. Analytical expressions for the velocity field, the tangential drag and the work done by the drag force have been obtained and the corresponding Newtonian cases deduced. In obtaining the analytical solutions for the velocity components, it was assumed that they had the frequencies of the velocities of the corresponding boundary components. Some numerical work is done and comparisons made with Newtonian fluids so as to get some insight into the effects of elasticity.

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GOVERNING EQUATIONS

Using the rheological equation of state for the upper convected Maxwell fluid is given by:

$$\underset{\sim}{T} = -p\underset{\sim}{I} + \underset{\sim}{S} \tag{1}$$

$$\underset{\sim}{S} + \lambda \overset{\vee}{\underset{\sim}{S}} = 2\mu \underset{\sim}{D} \tag{2}$$

Where:

$\underset{\sim}{T}$ = The total stress.
$\underset{\sim}{S}$ = The extra stress tensor.
$\underset{\sim}{D}$ = The deformation rate tensor.
p = An isotropic pressure.
λ = The relaxation time.
μ = The viscosity coefficient.
" $\vee$ " = Represents the upper-convected derivative, defined by:

$$\overset{\vee}{\underset{\sim}{S}} = \overset{\vee}{S}^{ij} = \frac{\partial S^{ij}}{\partial t} + q^m S^{ij}{}_{,m} - S^{im} q^j{}_{,m} - S^{mj} q^i{}_{,m} \tag{3}$$

The unsteady flow of the viscoelastic, incompressible upper convected Maxwell fluid is characterised by the longitudinal and torsional oscillations of the cylindrical rod with velocity,

$$\underset{\sim}{q}_b = q_0 \cos(\Omega_1 t)\cos(\beta)\hat{\theta} + q_0 \cos(\Omega_2 t)\sin(\beta)\hat{z} \tag{4}$$

Where:

q_0 = The magnitude of the oscillations.
Ω_1 = The frequency of the torsional oscillation.
Ω_2 = The frequency of the longitudinal oscillation.
β = Can be interpreted as the angle which the boundary velocity, $\underset{\sim}{q}_b$ makes with the $\hat{\theta}$-direction.

It should be noted that if $\beta = 0$ the cylinder has purely torsional oscillations and when $\beta = \pi/2$ the oscillations are purely longitudinal.

The dynamic equation is:

$$\nabla \bullet \underset{\sim}{S} - \nabla p = \rho \frac{d\underset{\sim}{q}}{dt} \tag{5}$$

while, the continuity equation is,

$$\nabla \bullet \underset{\sim}{q} = 0 \tag{6}$$

Working in cylindrical polar co-ordinates (R, θ, z), with the z-axis coinciding with the axis of the cylinder, we take the pressure field to be independent of the $\hat{\theta}$ and $\hat{z}$ co-ordinates and the velocity field to be of the form:

$$\underset{\sim}{q}(R,t) = v(R,t)\,\hat{\theta} + w(R,t)\,\hat{z} \tag{7}$$

Substituting Eq. (7) into the constitutive and dynamic Eq. 2 and 5, leads to the following equations,

$$S_{R\theta} + \lambda \frac{\partial S_{R\theta}}{\partial t} = \mu\left(\frac{\partial v}{\partial R} - \frac{v}{R}\right) \tag{8}$$

$$S_{Rz} + \lambda \frac{\partial S_{Rz}}{\partial t} = \mu \frac{\partial w}{\partial R} \tag{9}$$

$$\frac{2}{R} S_{R\theta} + \frac{\partial S_{R\theta}}{\partial R} = \rho \frac{\partial v}{\partial t} \tag{10}$$

$$\frac{1}{R} S_{Rz} + \frac{\partial S_{Rz}}{\partial R} = \rho \frac{\partial w}{\partial t} \tag{11}$$

$$S_{\theta\theta} + \lambda\left(\frac{\partial S_{\theta\theta}}{\partial t} - 2S_{R\theta}\frac{\partial v}{\partial R}\right) = 0 \tag{12}$$

$$\frac{\rho}{R} v^2 = \frac{\partial p}{\partial R} + \frac{1}{R} S_{\theta\theta} \tag{13}$$

Eliminating the stresses from Eq. (8) and (10) gives the equation of motion for the $\hat{\theta}$-component of the velocity field,

$$\frac{\partial v}{\partial t} = \upsilon\left(\frac{\partial^2 v}{\partial R^2} + \frac{1}{R}\frac{\partial v}{\partial R} - \frac{v}{R^2}\right) - \lambda \frac{\partial^2 v}{\partial t^2} \tag{14}$$

Similarly, using Eq. (9) and (11), the equation of motion for the $\hat{\theta}$-component is,

$$\frac{\partial w}{\partial t} = \upsilon\left(\frac{\partial^2 w}{\partial R^2} + \frac{1}{R}\frac{\partial w}{\partial R}\right) - \lambda \frac{\partial^2 w}{\partial t^2} \tag{15}$$

By solving Eq. (14) and (15), the velocity components can be determined. It should be noted that setting the elastic parameter $\lambda = 0$ gives the governing equations for classical fluids. Once the velocity field is determined the pressure field can be found using the Eq. 10, 12 and 13.

STATEMENT AND SOLUTION OF THE PROBLEM

Consider a long, straight, solid, circular, cylindrical rod, of uniform cross section, of radius a, oscillating both longitudinally and torsionally with different frequencies, in an Upper Convected Maxwell fluid. In the analysis of this problem, it was assumed that the rod was infinite in length, the flow was fully established, the fluid is at rest at infinity and there were no external forces acting on the rod.

We need to examine the flow of the fluid subject to the following kinematic conditions,

$$\underset{\sim}{v}(a,t) = q_0 \cos(\Omega_1 t)\cos(\beta)\,\hat{\theta} \tag{16}$$

$$\underset{\sim}{w}(a,t) = q_0 \cos(\Omega_2 t)\sin(\beta)\,\hat{z} \tag{17}$$

$$\underset{\sim}{v}(R,t) \to 0 \text{ as } R \to \infty \tag{18}$$

$$\underset{\sim}{w}(R,t) \to 0 \text{ as } R \to \infty \tag{19}$$

Note that on substituting Eq. (7) into (6) gives the continuity equation being automatically satisfied.

Assuming that v (R, t) and w (R, t) are of the form $\Re[f(E)e^{i\Omega_j t}]$, where, j = 1, 2, respectively for v (R, t), w (R, t) and '$\Re$' represents the real part of the expression, the solutions of Eq. 14 and 15 subject to 16-19 are,

$$v(R,t) = \Re\left[\frac{K_1(\gamma_1 R)}{K_1(\gamma_1 a)}e^{i\Omega_1 t}\right]q_0\cos(\beta) \tag{20}$$

and

$$w(R,t) = \Re\left[\frac{K_0(\gamma_2 R)}{K_0(\gamma_2 a)}e^{i\Omega_2 t}\right]q_0\sin(\beta) \tag{21}$$

where,

$$\gamma_j = \sqrt{\left(\frac{i\Omega_j - \Omega_j^2\lambda}{\upsilon}\right)} \tag{22}$$

and $K_n(x)$ is the modified Bessel function (Watson, 1952), of the second kind of order n.

The velocity field has therefore been determined. Note, on taking the elastic parameter $\lambda = 0$, the result reduces to the Newtonian case (Casarella and Laura, 1969). With the aid of Eq. 20 and 21 and on utilizing 8 and 9, the shear stresses on the cylinder are found to be,

$$S_{R\theta} = -\Re\left[\frac{\gamma_1 K_0(\gamma_1 a) + \frac{2}{a}K_1(\gamma_1 a)}{K_1(\gamma_1 a)}\left(\frac{e^{i\Omega_1 t}}{1+i\Omega_1\lambda}\right)\right]\mu q_0\cos(\beta) \tag{23}$$

and

$$S_{Rz} = -\Re\left[\frac{\gamma_2 K_1(\gamma_2 a)}{K_0(\gamma_2 a)}\left(\frac{e^{i\Omega_2 t}}{1+i\Omega_2\lambda}\right)\right]\mu q_0\sin(\beta) \tag{24}$$

The tangential drag $\underset{\sim}{D}$ acting on the cylinder per unit length is (Casarella and Laura, 1969):

$$\underset{\sim}{D} = -2\pi a(S_{R\theta}\hat{\theta} + S_{Rz}\hat{z})|_{R=a} \tag{25}$$

Substituting Eq. (23) and (24) into Eq. (25) gives,

$$\underset{\sim}{D} = 2pa\mu q_0\Re\begin{bmatrix}\left(\frac{\gamma_1 K_0(\gamma_1 a)}{K_1(\gamma_1 a)} + \frac{2}{a}\right)\frac{e^{i\Omega_1 t}}{1+i\Omega_1\lambda}\cos(\beta)\hat{\theta} + \\ \left(\frac{\gamma_2 K_1(\gamma_2 a)}{K_0(\gamma_2 a)}\right)\frac{e^{i\Omega_2 t}}{1+i\Omega_2\lambda}\sin(\beta)\hat{z}\end{bmatrix} \tag{26}$$

The work done, W_j, by the drag force, $\underset{\sim}{D}$ on the fluid per half-cycle of torsional and longitudinal motion is given by:

$$W_j = -\int_0^{\frac{\pi}{\Omega_j}} \underset{\sim}{D}\bullet\underset{\sim}{q_b}\,dt \tag{27}$$

where, j = 1, 2 refer to the torsional and longitudinal motions, respectively.

Substituting Eq. 4 and 26 into Eq. 27 gives,

$$W_j = -\pi a q_0^2\mu\Re\begin{bmatrix}\left(\frac{\gamma_1 K_0(\gamma_1 a)}{K_1(\gamma_1 a)} + \frac{2}{a}\right)\frac{\cos^2(\beta)}{1+i\Omega_1\lambda}\hat{I}(\Omega_j,\Omega_1) + \\ \left(\frac{\gamma_2 K_1(\gamma_2 a)}{K_0(\gamma_2 a)}\right)\frac{\sin^2(\beta)}{1+i\Omega_2\lambda}\hat{I}(\Omega_j,\Omega_2)\end{bmatrix} \tag{28}$$

where,

$$\hat{I}(\Omega_j,\Omega) = 2\int_0^{\frac{p}{\Omega_j}} e^{i\Omega t}\cos(\Omega t)dt = \frac{\Omega_j\cos\left(\frac{\pi\Omega}{\Omega_j}\right)\sin\left(\frac{\pi\Omega}{\Omega_j}\right) + \pi\Omega - i\Omega_j\cos^2\left(\frac{\pi\Omega}{\Omega_j}\right) + i\Omega_j}{\Omega\Omega_j} \tag{29}$$

NUMERICAL RESULTS AND CONCLUDING REMARKS

In order to investigate the effects of elasticity, comparisons between the U.C.M. and Newtonian fluids

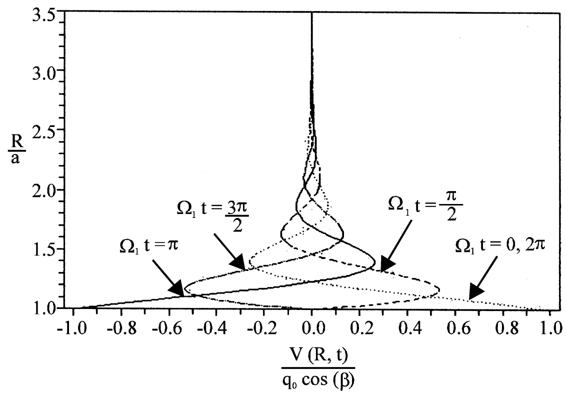

Fig. 1: Behaviour of the $\hat{\theta}$-component of velocity for the U.C.M. fluid

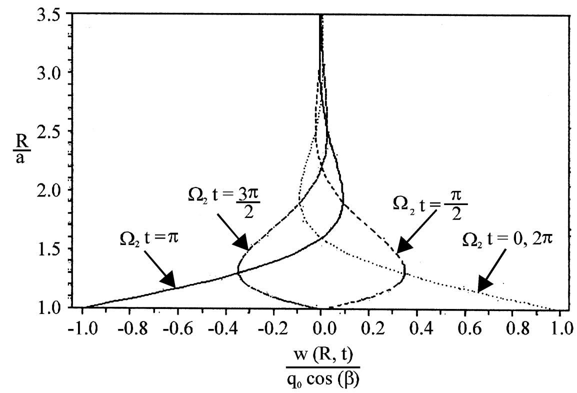

Fig. 3: Behaviour of the $\hat{z}$-component of velocity for the U.C.M. fluid

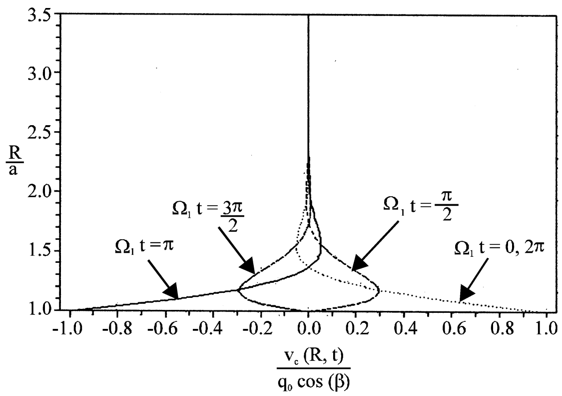

Fig. 2: Behaviour of the $\hat{\theta}$-component of velocity for the Newtonian fluid

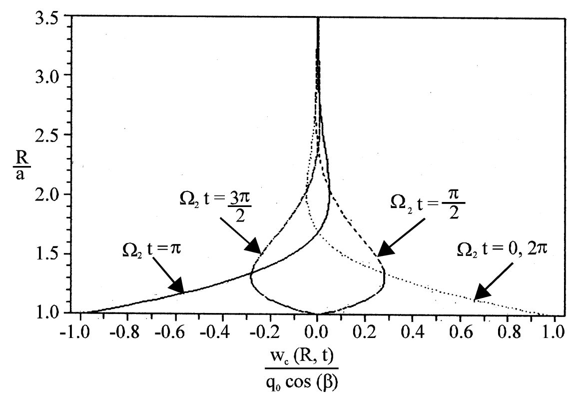

Fig. 4: Behaviour of the $\hat{z}$-component of velocity for the Newtonian fluid

were made for the velocity components and the work done by the drag force, for the different oscillation frequencies Ω_1 and Ω_2. Here, the values taken are, a = 1, $\upsilon = 0.1$ and $\lambda = 0.3$, these are the same as those selected by Rahaman (2005) and Owen and Rahaman (2006) the frequencies considered were $\Omega_1 = 3.6$ and $\Omega_2 = 1$.

The behaviour of the velocity profiles of the $\hat{\theta}$-component are displayed in Fig. 1 and 2 for the U.C.M. and Newtonian fluids respectively. It can be observed that as one moves away from the cylindrical rod, the $\hat{\theta}$-component is damped out quicker for Newtonian fluids than that of U.C.M. fluids. Also, the U.C.M. flow is reversed many more times than that of its corresponding Newtonian case.

From Fig 3 and 4, it is also observed that the behaviour of the $\hat{z}$-component for the Newtonian fluid appears to be damped out quicker that than of the U.C.M. fluid, but not as drastic as that observed for the $\hat{\theta}$-component. These velocity profiles are symmetric about axes of zero velocity and in particular, symmetry is observed at times t = 0, π/Ω_i and $t = \pi/2\Omega_i$, $3\pi/2\Omega_i$.

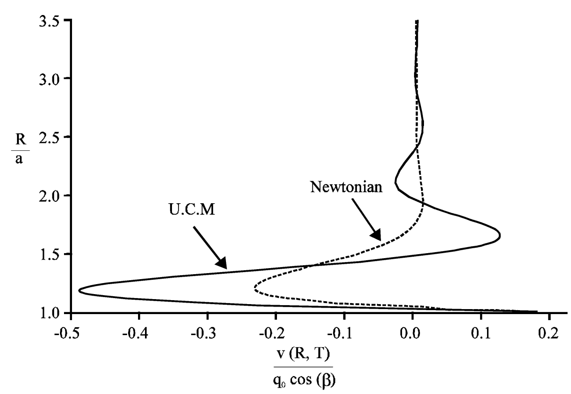

Fig. 5: Behaviour of the $\hat{\theta}$-component of velocity at t = T for the Newtonian and the U.C.M. fluids

Figure 5 and 6 show the behaviour of the velocity profiles for the $\hat{\theta}$ and $\hat{z}$-components at a particular time

$$t = T = \frac{\pi}{\frac{(\Omega_1 + \Omega_2)}{2}}$$

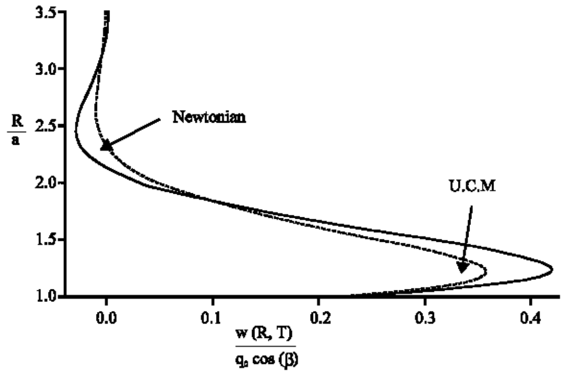

Fig. 6: Behaviour of the $\hat{z}$ -component of velocity at t = T for the Newtonian and U.C.M. fluids

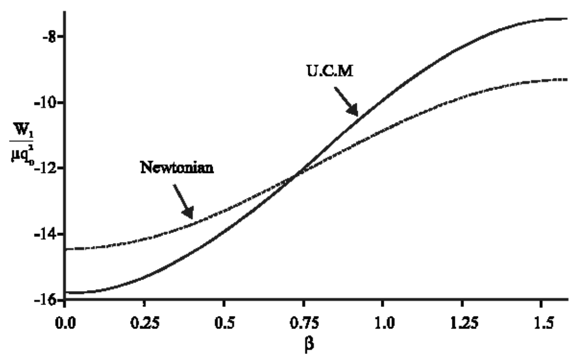

Fig. 7: Behaviour of the work done by the drag force per unit half cycle of the torsional motion for Newtonian and U.C.M. fluids

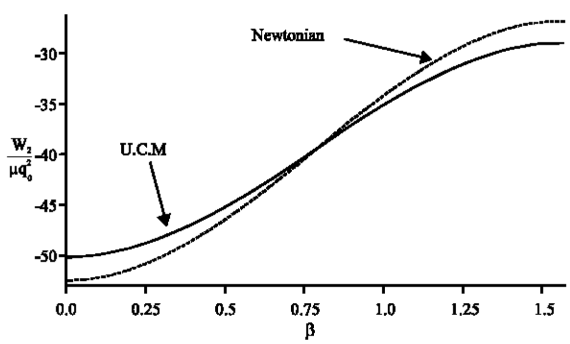

Fig. 8: Behaviour of the work done by the drag force per unit half cycle of the longitudinal motion for Newtonian and U.C.M. fluids

as considered by Owen and Rahaman (2006). In both cases, it can be seen that the components are damped out quicker for the Newtonian fluids than that of the U.C.M. fluids. Flow reversal is observed for each component.

Figure 7 and 8 display the behaviour of the work done by the drag force per unit half cycle of the torsional and longitudinal motion respectively, as β varies from 0 to $\pi/2$. For the torsional motion, the magnitude of work done for the U.C.M. fluid is initially larger than that of the classical fluids, however, this changes as β increases. For the longitudinal motion, the magnitude of work done for the U.C.M. fluid is initially smaller than that of the classical fluids but this also changes as β increases.

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