Journal of Engineering and Applied Sciences 3 (7): 569-573, 2008
ISSN: 1816-949X

A Quadratic Programming Model for Selection of Fertilizer Combination for Crop Optimal Yield Using the Concept of Response Surface Methodology

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Abstract: In every day life resources are lost because of human inability to transform most of our typical life problems into statistical or mathematical concept. Over the years agriculturist have lost so much in the purchase of different types of fertilizers and yet cannot combine them to obtain optimal yield of their crops. A quadratic programming model for this problem is here established using the concept of response surface methodology. Investigations shows that the model suit the Nigerian situation since it prevents sub-optimality some times obtained by farmers which may be caused by wrong combination of fertilizers possibly based on tradition and customs.

Key words: Quadratic programming, response surface, optimal design, transformation

INTRODUCTION

Response surface methodology is practically a new area in experimental designs, its applicability cut across agriculture, operation research and many other areas of human endeavours.

The concept was first originated by Box and Wilson (1954), who in comparing the performance of some experimental design coined the word response surface, its rule in Agricultural Context cannot be over-emphasized, Mela (1986) developed a model for inter-cropping of different species of crops using the concept of response surface designs though the result gotten was not consistent because, the statistical tools used in generating the result was a little bit complicating.

There are so many research work on this area but some of the outstanding ones are that of Lemke (1952), in his study of productive efficiency of Fifian system in semi subsistence agriculture. Jolayemi (1996) also developed a quadratic optimization model for selecting crops for mixed cropping, he extended Lemke's product mix model to incorporate the interaction effect of crops grown together.

The model involves maximization of profit subject to constraint such as land and cash availability to differ in operations like procurement of seedlings and fertilizer, he developed 3 different solution procedures such as optimization with respect to every possible group of crops technique based on the magnitude of sub groups, net total profits and technique based on systematic exclusion of variable.

This study set to deviate a little from the general opinion of many authors, in the area and went ahead to establish a new method of formulating the objective function and the constraints by looking at the expected yield when different fertilizer combinations are applied to these crops and the cost involved in purchasing these fertilizer types, this is subject to availability of cash and other exigencies which posed the constraints.

Reference model: Hillier and Lieberman (1974) is chosen as the reference model in this case. Here the objective function which is quadratic in nature is maximized subject to Linear constraints.

$$\text{Max}(Z) = \sum \alpha_j \beta_k + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} q_{jk}\gamma_j\gamma_k \qquad (1)$$

Subject to $\Sigma a_i X_j \leq b_i$ for i = 1, 2, --- m and $X_j \geq 0$ for j = 1, 2, … n, where, a_{ij}, b_i α_j and q_{jk} are known constants.

The function being maximized is called the objective function, while the restrictive

$$\sum_{i=1}^{n} a_i x_j \leq b_i$$

are called functional constraints. Hillier and Lieberman (1974) has shown that any maximization of quadratic programming problem can be converted to a minimization problem by multiplying the function by -1,

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thus (1-1) can be converted to a minimization problem and stated in matrix form as follows:

$$\text{Min } f(x) = ½\, x^T + G_x + g_x{}^t \text{ S.T} \qquad (2)$$

$$A^T x = b$$

It is assumed that there are $M \le n$ constraints so that $b \in R^m$. A is an nxm matrix, $a_i \in A$, LEE and A is of rank m.

Using the constraints to eliminate variable in Eq. (2) the model can now be obtained thus,

Let the partitions

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

be defined where, $X_1 \in R^m$ and $X_2 \in R^{n-m}$ etc, then the constraints now becomes

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^T \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = b$$

$$\begin{pmatrix} A_1^T & A_2 \end{pmatrix}^T \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = b$$

$$A^T{}_1 X_1 + A^T{}_2 X_2 = b$$

$$A^T{}_1 X_1 = b - A^T{}_2 X_2$$

$$X_1 = A_1{}^{-T} (b - A^T{}_2 X_2)$$

Substituting into the objective function we have

$$\psi(x_2) = ½ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^T \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}^T \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$= ½\, (x_1 \;\; x_2) \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} g_1 & g_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$= ½\, (X_1\; G_{11} + X_2\; G_{22}\; X_1\; G_{12} + X_2\; G_{22} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + g_1^T X_1 + g_2^T X_2$$

$$= ½\, X_1{}^T G_{11} X_1 + x_2\, G_{21} X_1 + X_1 G_{12} X_2 + X_2{}^T G_{22} X_2 + g_1{}^T X_1 + g_2{}^T X_2$$

Thus, ψX_2 now becomes

$$\begin{aligned}
& ½\, [A_1{}^{-T} (b - A_2{}^T X_2)\, G_{11}\, A_1{}^{-T} (b - A_2{}^T X_2) + X_2\, G_{21}\, A_1{}^{-T} (b - A_2{}^T X_2) \\
& + A_1{}^{-T} (b - A_2{}^T X_2 + G_{12} X_2 + X_2{}^T\, G_{22} X_2)] + g_1{}^T A_1{}^{-T} (b - A_2{}^T X_2) + g_2{}^T X_2 \\
& = ½\, [(A_1{}^{-T} b - A_1{}^{-T} A_2{}^T X_2)(G_{11} A_1{}^{-T}\, b - G_{11} A_1{}^{-T} A_2{}^{-T} X_2) + X_2 G_{21} A_1{}^{-T} \\
& \quad b - X_2\, G_{21}\, A_1{}^{-T}\, A_2{}^{-T} A_2{}^T X_2 + A_1{}^{-T}\, b G_{12}\, x_2 - A_1{}^{-T}\, A_2{}^T X_2 G_{12} X_2 \\
& + X_2{}^T\, G_{22}\, X_2] + g_1{}^T\, A_1{}^{-T}\, b - g_1{}^T A_1{}^{-T} A_2{}^T X_2 + g_2{}^T\, X_2 \\
& = ½\, [A_1{}^{-T}\, b G_{11}\, A_1{}^{-T}\, b\text{-}A_1{}^{-T}\, b G_{11} A_1{}^{-T}\, A_2{}^T X_2 - A_1{}^{-T} A_2{}^T X_2 G_{11} A_1{}^{-T} b + \\
& \quad A_1{}^{-T} A_2{}^T X_2{}^{G}{}_{11}\, A_1{}^{-T}\, A_2{}^T X_2{}^{+X}{}_2 G_{21}\, A_1{}^{-T} b - X_2 G_{21}\, A_1{}^{-T} A_2{}^T X_2 + A_1{}^{-T} \\
& \quad b G_{12}\, X_2 - A_1{}^{-T}\, A_2{}^T X_2\, G_{12}\, X_2 + X_2{}^T G_{22} X_2]\; g_1{}^T A_1{}^{-T} b - g_1{}^T A_1{}^{-T} A_2{}^T X_2 + g_2{}^T X_2 \\
& = ½\, X_2{}^T\, (G_{22} - G_{21} A_1{}^{-T} A_2 - A_2 A^{-T}\, G_{12} + A_2 A_1{}^{-T} G_{12} + A_2 A_1{}^{-T} G_{11}\, A_1{}^{-T} A_2) \\
& + X_2{}^T\, (G_{21} - A_2 A_1{}^{-T} G_{11})\, A_1{}^{-T} b + ½\, b + A_1{}^{-T} G_{11}\, A_1{}^{-T} b + X_2{}^T \\
& \quad (g_2 + A_2 A_1{}^{-T} g_1) + g_1{}^T\, A_1{}^{-T} b
\end{aligned}$$

If $G > 0$, then the unique minimizer $X_2{}^*$ exist and it is obtained by solving the system

$$\nabla \psi(x_2) = 0.$$

DEVELOPMENT OF THE QUADRATIC PROGRAMMING MODEL

The expected yield when fertilizer combination X_j j = 1, … 4 are applied to crop j is $C_j X_j$. Therefore, the total expected yield that would be realized by applying each one of a set of n fertilizer simply on the crops is:

$$\sum_{j=1}^{n} C_j X_j \qquad (3)$$

If X_j, j = 1 … 4 are the decision variables, respectively for the different types of fertilizer combination for the 4 crops and if the interaction effect of this in term of optimal yield is $W_j k$. Then the total effect of interaction of these fertilizers in terms of crop optimal yield is:

$$\sum_{j=1}^{n}\sum_{k=1}^{n} W_j\ kx_j\ x_k \tag{4}$$

Therefore, the objective function for the fertilizer combination is:

$$\text{Max } f(x) = \sum_{j=1}^{n} C_j X_j + \tfrac{1}{2}\sum_{j=1}^{n}\sum_{j=1}^{n} W_{jk}\ X_j\ X_k \tag{5}$$

Subject to, the following constraints.

Cost of purchasing fertilizer type 1 for crop j is $X_j\ h_j$. Thus, the cost of purchasing fertilizer type 1 for a set of n crop is:

$$\sum_{j=1}^{n} X_j h_j,$$

but since the total cost available for purchasing fertilize type 1 is H.
Then, the first constraint is

$$\sum_{j=1}^{n} X_j h_j \le H. \tag{6}$$

The second constraint can thus be formed the same way, the cost of purchasing fertilizer type II for crop j is $g_j\ x_j$, thus the total cost of purchasing fertilizer II for a set of n crops is $\Sigma g_j\ X_j$, but the total cost available for the purchase of this type of fertilizer is G, then the second constraint is:

$$\sum_{j=1}^{n} g_j X_j \le G \tag{7}$$

Thirdly, the cost of purchasing fertilizer type III for crop j is $V_j\ X_j$, thus the total cost of purchasing fertilizer type III for a set of n crops is $\Sigma V_j\ X_j$, but the total fund available for it is V. Hence, the third constraint is:

$$\Sigma V_j\ X_j \le V. \tag{8}$$

Also, the cost of purchasing fertilizer type IV for crops j is $X_j\ t_j$, thus for a set of n crops, the total cost of purchasing fertilizer type IV is $\Sigma X_j\ t_j$, but the available fund for thus is T, hence the forth constraint is:

$$\Sigma X_j\ t_j \le T. \tag{9}$$

Now the cost of hiring labour from planting to harvesting w.r.t crop j is $M_j\ X_j$, but the total cost of hiring labour from planting to harvesting of n crops is $\Sigma M_j\ X_j$, but the total fund available for hiring labour from planting to harvesting of crop n is M, hence, the constraint is:

$$\sum_{j=1}^{n} M_j X_j \le M \tag{10}$$

Also, the cost of purchasing crop type for the entire hectare of land is $\Sigma f_j\ X_j$, but the fund available is F, thus the sixth constraint is:

$$\Sigma f_j\ X_j \le f. \tag{11}$$

The cost of insurance coverage w.r.t crop j is $X_j\ l_j$, the total cost of insurance coverage for the set of n crop for the entire hectare of land is $\Sigma X_j\ l_j$ and the available funds for this is I. The seventh constraint is:

$$\Sigma X_j\ i_j \le I \tag{12}$$

Finally, the cost of post harvesting w.r.t. crop j is $X_j\ k_j$ and for the entire farm land we will have $\Sigma X_j\ k_j$, but the total fund available for this is k; hence the eigth constraint is:

$$\Sigma X_j\ k_j \le k \tag{13}$$

Combining Eq. (3-13) we have the quadratic programming model for selection of fertilizer combination for crop optimal yield to be

$$\text{Max } f(x) = \sum_{j=1}^{n} C_j x_j + \tfrac{1}{2}\sum_{j=1}^{n}\sum_{j=1}^{n} W_{jk}\ x_j x_k$$

$$\text{S.t. } \sum_{j=1}^{n} h_j X_j \le H$$

$$\sum_{j=1}^{n} g_j x_j \le G$$

$$\sum_{j=1}^{n} V_j x_{jj} \le T$$

$$\sum_{j=1}^{n} f_j x_j \le F$$

$$\sum_{j=1}^{n} \ell_j x_j \le I$$

$$\sum_{j=1}^{n} k_j x_j \le K$$

$$X_j \leq A, j = 12 \ldots n$$

$$X_j \geq 0, j = 12 \ldots n.$$

Theorem: Quadratic programming problems in canonical form can be converted into linear complementability if they are in dual and primal form.

Proof: Let's consider a linear programming problem P^L in canonical form and its dual P_D^L.

$$P^L : \text{Max } Cx$$

$$\text{S.t } Ax \leq b$$

$$X \geq 0$$

Or in its primal form we have

$$P^L_D \text{ Min } ub$$

$$\text{S.t } UA \geq C$$

$$U \geq 0$$

Based on P^L and P_D^L the following linear system can be established.

$$P^X : \begin{array}{l} AX \leq b \\ X \geq O \\ X \leq O \\ UA \geq O \\ U \geq O \\ Cx - Ub = O \end{array}$$

Note: $(\bar{X}, \bar{U})$ is a solution to system P^X if $\bar{X}$ is an optimal solution to P^L and $\bar{U}$ is an optimal solution to P_D^L. By introducing a vector of non negative slack variable y such that Ax + y = b and a vector of non negative variable v such that UA-V = C.

The constraint CX – Ub = 0 becomes (UA – U)X – U (AX + y) = 0 or VX + Uy = 0 so that the system P^X is transformed into

$$\begin{array}{c} Ax + y = b \\ UA - V = c \\ Ux + dy = o \\ X, U, y, V = 0 \end{array}$$

Given MZ + q = W and assuming m and q are given (n×n) and (n×1) dimensional matrices, respectively, then the linear complementarity is then to find vector w and z such that $Z^TW = 0$

$$Z, W \geq 0$$

But by definition

$$M = \begin{vmatrix} 0 & -A \\ A^T & 2Q \end{vmatrix}$$

$$Q = \begin{bmatrix} C^T \\ b \end{bmatrix} \quad Z = \begin{bmatrix} X \\ U^T \end{bmatrix}$$

and

$$W = \begin{bmatrix} V^T \\ y \end{bmatrix}$$

The complementarity condition is achieved if $Z^TW = 0$, but

$$Z^TMZ = \begin{bmatrix} U, X^T \end{bmatrix} \begin{bmatrix} X & -A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} U^T \\ x \end{bmatrix}$$

$$= \begin{bmatrix} X^TA^T & -UA \end{bmatrix} \begin{bmatrix} U^T \\ x \end{bmatrix}$$

$$= X^TA^TU^T - Uax$$

$$= (UA_x)^T - Uax = 0$$

$$\text{but } (UA_x)^T = UAx \Rightarrow UAx - Uax = 0$$

Hence, the complementarity condition is achieved.

CONCLUSION

The quadratic programming model for selection of fertilizer combination for crop optimal yield developed by us suit the Nigerian situation since it prevent sub-optimality.

The model is recommended for use by all farmers who may want to combine 2-5 types of fertilizers to obtain optimal yield of their crop. The analysis carried out by us show that the model work well when placed under certain precautions like planting of crops under equal spacing distance and application of uniform dosage of fertilizer at the initial stage and the ability to stop immediately the crop show first maturity.

These, if strictly followed will definitely manifest which fertilizer combination is good for a particular crop at harvest period.

Definition of variables:

β = Number of hectare available for planting.
ψ = Total number of fertilizer to be combined for n crops.
μ = Total number of crops to be combined for the Q fertilizer.
l_j = Cost of preparation of land with respect to crop j, j = 1, 2, ..., n.
f_j = Cost of purchasing seedling per hectare w.r.t. crop j.
S_j = Cost of purchasing fertilizer needed per hectare of land for crop j, j = 1, 2, ...,n.
M_j = Cost of hiring labour from planting to harvesting w.r.t. crop j.
h_j = Cost of purchasing fertilizer type 1 for crop j.
g_j = Cost of purchasing fertilizer type 2 for crop j.
V_j = Cost of purchasing fertilizer type 3 for crop j.
t_j = Cost of purchasing fertilizer type 4 for crop j.
e_j = Expected yield per crop when different fertilizer combination is applied to crop j.
k_j = Cost of post harvest handling per hectare of land.
S_j = Cost of insurance coverage per hectare of land.
H = Total fund available for purchase of fertilizer type 1.
G = Total fund available for purchase of fertilizer type II.
K = Total fund available for purchase of fertilizer type III.
T = Total fund available for purchase of fertilizer type IV.
S = Total fund available for insurance coverage.
K = Total fund available for post harvesting.
M = Total cost available for hiring labour.
W_{jk} = The interaction effect between fertilizer combination on crop j, j = 1, 2, ... k.
A = The size of the entire farm land.

Decision variables:

X_1 = Decision variable representing fertilizer combination allocated to crop J, j = 1.
X_2 = Decision variable representing fertilizer combination allocated to crop j, j = 2.
X_3 = Decision variable representing fertilizer combination allocated to crop j, j = 3.
X_4 = Decision variable representing fertilizer combination allocated to crop j, j = 4.

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