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A New Approach for the Derivation of Higher-Order Newton-Cotes Closed Integration Formulae

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Abstract: This study concerns the derivation of higher order Newton-cotes closed integration using Lagrange's interpolation polynomial functions. In literature, the conventional way of deriving these integration formulae is by making the x_i equally spaced, so that there are only n+1 parameters a_i to choose. The coefficients are then fixed either by using finite difference results or by considering it as a general formula with various coefficients which must be fixed. William presents coefficients for orders p = 1, 2,...8 . In this research, an extension of this was carried out for orders p = 9, 10, 11. Ralston gave a detailed mathematical treatment of the error analysis for the known Newton Cotes formulae of orders p=1,2,…,8. These manipulations are extremely cumbersome since it involves n+1 conditions and hence solving a large system of algebraic equations. This led to the new approach of using the Lagrange's interpolation functions to derive the Newton Cotes Closed formulae of orders p = 1,2,…,11. A generalized formula was developed for order p=k. These methods were applied to some integrals such as exponential function and trigonometric functions. The numerical results show that higher order methods are needed for equally spaced intervals to increase the accuracy of the general Newton-cotes closed formulae.

Key words: Newton-cotes, integration formulae, Lagrange's interpolation, polynomial functions, finite differences, closed formulae

INTRODUCTION

The basic approach to analytic integration is to see the linear operator as a summation of the area under a given curve or function.

Numerical integration is a faster approach and a more reliable approach to obtain numerical values for the integral at each node. We shall discuss in this study the "closed" formulae with x_i = a+ih; i = 0(i)n and h = b-a/n. The term "closed" implies that in all the subintervals, both the end-points are nodes of the integration formula.

The Newton-Cotes formulae are extremely useful and straightforward family of numerical integration Techniques. Numerical methods started due to the nasty nature of some integrals, which cannot be expressed in terms of familiar functions. In Electrical Field Theory, it is proved that the magnetic field induced by a current flowing in a circular loop of wire has intensity

$$H(x) = \frac{4Ir}{r^2 - x^2}\int_0^{\frac{\pi}{2}}\left[I - \left(\tfrac{x}{r}\right)^2 \mathrm{Sin}^2\theta\right]^{\frac{1}{2}} d\theta \qquad (1)$$

Where, I is the current, r is the radius of the loop and x is the distance (0≤x≤r). This is an elliptic integral and not expressible in terms of familiar functions. Only a numerical method can easily give a precise result. This is also applicable in most complex engineering problems.

Ralston (1965) gave a detailed mathematical treatment of the error analysis for the known Newton Cotes formulae of orders p=1,2,…,8. These manipulations are extremely cumbersome since it involves n+1 conditions and hence solving a large system of algebraic equations. Unlike numerical differentiation, numerical integrations are stable and more accurate with appropriate choice of the interpolating polynomials. Can we get a better way of deriving higher order Newton-Cotes closed formulae for the integration of such integrals? This and other questions motivated this project.

SOME IMPORTANT RESULTS

The first two members of the Newton-Cotes Closed formulae are the Trapezoid rule and the Simpson's methods.

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We state here two known results on the precision of these two integrations.

Theorem 1: If f″ exists and is continuous on [a,b] and if the composite trapezoid rule T with uniform spacing "h" is used to estimate the integral

$$I = \int_a^b f(x)dx$$

then for some a, b ∈ R,

$$I - T = -\frac{1}{12}(b-a)h^2 f''(\xi) = 0(h^2) \quad (2)$$

Theorem 2: Suppose that

$$I = \int_a^b f(x)dx$$

is estimated by the Simpson's rule using N equal subdivisions of [a, b] and $f^{(iv)}$ is continuous, then the error term ($e_{iv}(x)$) resulting from this approximation is given by

$$e_{iv}(x) = -\frac{b-a}{180}h^4 f^{(iv)}(\theta) \text{ forsome } \theta \in [a,b] \text{ whrer, } h = \frac{b-a}{2N}$$

Beyer (1987) presents some higher order Newton-Cotes rules

$$\int_{x_1}^{x_n} f(x)dx = h\left(\frac{2}{5}f_1 + \frac{11}{20}f_2 + f_3 + \ldots + f_{n-2} + \frac{11}{20}f_{n-1} + \frac{2}{5}f_n\right) \quad (3)$$

$$\int_{x_0-3h}^{x_0+3h} f(x)dx = \frac{h}{100}(28f_{-3} + 162f_{-2} + 22f_0 + 162f_3 + 28f_3) + \frac{9}{1400}h^7\left(2f^{(4)}(\xi) - h^2 f^{(8)}(\xi)\right) \quad (4)$$

$$\int_{x_1}^{x_n} f(x)dx = \frac{3}{10}h(f_1 + 5f_2 + f_3 + 6f_4 + f_5 + 5f_6 + f_7) \quad (5)$$

as Duran's rule, Hardy's rule and Waddle's rule, respectively (Endre and Meyers, 2003).

Conventional approach to deriving newton-cotes closed formulae: The Newton-Cotes formulae approximates the integral

$$I = \int_a^b f(x)dx$$

by integrating a polynomial $Q_n(x_i)$ of degree n whose coefficients are chosen so that

$$Q_n(x_i) = f(x_i), \quad i = 0, 1, 2, \ldots, n \quad (6)$$

The closed formulae are considered when

$$x_i = a + ih, \quad i = 0, 1, 2, \ldots, n; h = (b-a)f_n \quad (7)$$

Consider the following cases of approximation and the evolving integration formulae!

Case I: Approximation by a polynomial of degree zero

$$Q_0(x) = c_0 \Rightarrow I = \int_a^b f(x)\,dx \cong \int_a^b Q_0(x)dx = hf_0 \quad (8)$$

Case II: Approximation by a polynomial of degree one

$$Q_1(x) = c_1 x + c_0 \Rightarrow I = \int_a^b f(x)\,dx \cong \frac{h}{2}[f_0 + f_1] \quad (9)$$

Case III: Approximation by a polynomial of degree 2

$$Q_2(x) = c_2 x^2 + c_1 x + c_0 \Rightarrow I = \int_a^b f(x)\,dx \cong \frac{h}{3}[f_0 + 4f_1 + f_2] \quad (10)$$

Note that Eq. 8-10 are the rectangle rule, trapezoidal and the h/3 Simpson's rule, respectively (Uerberhuber, 1997).

By taking a series of increasing order polynomials, a family of integration formulae will be generated of the generalised form

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f_i + E \quad (11)$$

Where, E is the error term is expected to be as small as possible.

Determination of the coefficients and error analysis: The general form of the approximation to the integral has 2n+2 variable coefficients i.e. a_i and x_i which must be chosen for optional accuracy of the formulae. In the case of Newton-Cotes formulae, some degree of choice has been sacrificed by making the x_i equally spaced, so that there now only n+1 parameters a_i to choose. There are n+1 conditions required to fix the coefficients a_i. These conditions are provided by making the formula exact for the polynomials $1, x, x^2, x^3, \ldots, x^n$.

As an illustrative example, we consider the closed Newton-Cotes Formula of order four (4) over the interval 0-4h.

$$\int_0^{4h} f(x)\,dx = \sum_{i=0}^{4} a_i f_i + E, \qquad x_i = 0 + ih \tag{12}$$

Since, the formula is exact for 1, x, x^2, x^3,..., x^n. and x^4, we substitute these functions in turn, into (12) giving 5 quations in 5 unknowns:

$$\left.\begin{aligned}
f(x) = 1 &\Rightarrow 4h = a_0 + a_1 + a_2 + a_3 + a_4 \\
f(x) = x &\Rightarrow 8h^2 = ha_1 + 2ha_2 + 3ha_3 + 4ha_4 \\
f(x) = x^2 &\Rightarrow \frac{64h^3}{3} = h^2a_1 + 4h^2a_2 + 9h^2a_3 + 16h^2a_4 \\
f(x) = x^3 &\Rightarrow \frac{256h^4}{4} = h^3a_1 + 8h^3a_2 + 27h^3a_3 + 64h^3a_4 \\
f(x) = x^4 &\Rightarrow \frac{1024h^4}{5} = h^4a_1 + 16h^4a_2 + 81h^4a_3 + 256h^4a_4
\end{aligned}\right\} \tag{13}$$

Resolving into matrix form,

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 9 & 16 \\ 0 & 1 & 8 & 27 & 64 \\ 0 & 1 & 16 & 61 & 256 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 4h \\ 8h \\ 64h/3 \\ 256h/4 \\ 1024h/5 \end{pmatrix} \tag{14}$$

Solving this system for a_i we obtain

$$a_0 = \frac{14h}{45}, \quad a_1 = \frac{64h}{45}, \quad a_2 = \frac{24h}{45}, \quad a_3 = \frac{64h}{45}, \quad a_4 = \frac{14h}{45} \tag{15}$$

Ralston (1965) and Turner (1991) show that the general form for the Newton-Cotes integration formulae including the error terms is

$$\int_a^b f(x)\,dx = A_0 \cdot h \sum_{i=0}^{n} w_i f_i + A_1 h^{k+1} f^{(k)}(\xi) \tag{16}$$

Where, n is the number of strips

$$h = \frac{b-a}{n}$$

and ξ is some value in the interval [a, b].

Following Ralston (1965) we have the results in Table 1.

Table 1: The general form for the Newton-Cotes integration formulae

n/p	A_0	w_0	W_1	w_2	w_3	w_4	A_1
½	1/2	1	1				-1/12
2/3	1/3	1	4	1			-1/90
3/4	3/8	1	3	3	1		-3/80
4/5	2/45	7	32	12	32	7	-8/945
5/6	5/288	19	75	50	50	75	-275/12096
6/7	1/140	41	216	27	272	27	-9/1400
7/8	7/17280	751	3577	1323	2989	2989	-8183/518400

DERIVATION TECHNIQUE USING LAGRANGE'S INTERPOLATION POLYNOMIAL FUNCTIONS

We present here an alternative approach to deriving the Newton Cotes closed formulae for the evaluation of special integrals. Here, we use the Lagrange's interpolation polynomials for the derivation in the spirit of Arnold and Christop(1998) as follows:

Consider the approximation for the integral

$$\int_a^b f(x)\,dx \tag{17}$$

Integrating a polynomial $P_n(x)$ of degree n, with coefficients chosen so that $P_n(x_i) = f(x_i)$;

$$I = 0, 1, ..., n \text{ and } h = \frac{b-a}{n}$$

The first member of these integration formulae using the Lagrange's interpolation polynomial is derived thus:

Taking the abscissas x_1 and $x_2 = x_i + h$, the Lagrange's interpolation polynomial through the points (x_1, f_1) and (x_2, f_2) is

$$P_2(x) = \left(\frac{x - x_2}{x_1 - x_2}\right) f_1 + \left(\frac{x - x_1}{x_2 - x_1}\right) f_2 = \frac{x}{h}(f_2 - f_1) + \left(f_1 + \frac{x_1}{h} f_1 - \frac{x_1}{h} f_2\right) \tag{18}$$

Integrating (18) over the interval $[x_1, x_1+h]$, we have

$$\int_a^b P_2(x)\,dx = \frac{1}{2h}(f_2 - f_1)\left[x^2\right]_{x_1}^{x_2} + \left(f_1 + \frac{x_1}{h} f_1 - \frac{x_1}{h} f_2\right)[x]_{x_1}^{x_2} = \frac{h}{2}(f_1 + f_2)\left[x^2\right]_{x_1}^{x_2} - \frac{h^3}{12} f''(\xi) \tag{19}$$

Equation 19 coincides with the conventional Trapezoidal rule with the error constant -1/12.

Following the same procedure for the Lagrange's interpolation polynomials of the order 2, 3,..., n we obtain the table of coefficients of the Newton-Cotes closed formulae for order p =1,..., n+1 (for max n = 10).

Table 2: Coefficients of the newly derived Newton-Cotes closed formulae of orders 2,3,…,11. using the Lagrange's Interpolation Functions

n/p	A_0	W_0	w_1	W_2	w_3	W_4	w_5	w_6	w_7	w_8	w_9	w_{10}	A_1
1/2	1/2	1	1										-1/12
2/3	1/3	1	4	1									-1/90
3/4	3/8	1	3	3	1								-3/80
4/5	2/45	7	32	12	32	7							-8/945
5/6	5/288	19	75	50	50	75	19						-275/12096
6/7	1/140	41	216	27	272	27	216	41					-9/1400
7/8	7/17280	751	3577	1323	2989	2989	1323	3577	751				-8183/518400
8/9	4/14175	989	5888	-928	10496	-4540	10496	-928	5888	989			-2368/467775
9/10	9/89600	2857	15741	1080	19344	5778	5778	19344	1080	15741	2857		-173/14620
10/11	5/299376	16067	106300	-48525	272400	-260550	427368	-260550	272400	-48525	106300	16067	-1346350/ 32691859280

n is the order of the polynomial, p is the order of the closed Newton-Cotes Integration Formulae, A_1, w is the *coefficient* of the closed Newton-Cotes Integration Formulae, A_0 is the Truncation Error constants of the closed Newton-Cotes Integration Formulae

Table 3: Absolute errors for methods of minimum and maximum orders

	Absolute errors for the various method	
Integral	Order 1	Order 11
$\int_{-1}^{1} \operatorname{Sin} x\, dx$	0	0
$\int_{0}^{0.8} \operatorname{Sin} x\, dx$	2.7×10^{-3}	5.0×10^{-3}
$\int_{0}^{1} \tan x\, dx$	4.4×10^{-3}	7.8×10^{-3}
$\int_{0}^{1} e^{x}\, dx$	1.40×10^{-1}	1.28×10^{-1}

For example, for n = 11, the abscissas are $x_{10} = x_1+9\,h$, $x_{11} = x_1 + 10\,h$ and the Lagrange's interpolation polynomial becomes

$$P_{11}(x) = \frac{(x-x_{10})(x-x_{11})}{(x_1-x_{10})(x_1-x_{11})} f_1 + \ldots\cdots + \frac{(x-x_1)(x-x_2)}{(x_{11}-x_1)(x_{11}-x_2)} f_{11} \tag{20}$$

$$\Rightarrow \int_{x_1}^{x_1+10h} f(x)\,dx = \frac{5h}{299376} \begin{pmatrix} 16067f_1 + 106300f_2 - 48525f_3 \\ +272400f_4 - 260550f_5 + 427368f \\ {}_6 -260550f_7 + 272400f_8 - 48525f_9 \\ +106300f_{10} + 16067f_{11} \end{pmatrix} - \frac{1346350}{326918592} h^{12} f''(\xi) \tag{21}$$

The n-order closed Newton-Cotes Formula is given by the analytic expression

$$\int_{x_1}^{x_n} f(x)\,dx = h\sum_{i=1}^{n} H_{n,\,r+1} \tag{22}$$

Where,

$$H_{n,\,r+1} = \frac{(-1)^{n-r}}{r!(n-r)!} \int_0^n t(t-1)\ldots(t-r+1)(t-r-1)\ldots(t-n)\,dt \tag{23}$$

So that

$$\sum_{r=0}^{n} H_{n,\,r+1} = n \tag{24}$$

DISCUSSION

Table 2 presents the coefficients of the newly derived Newton-Cotes closed formula using the Lagrange's interpolation polynomials approach. It is an easy and relatively simpler approach. It is interesting to note that all the coefficients are exactly the same with the ones derived using the conventional approach.

We apply these methods to handle some standard integrals and compare the results for the various orders shown in Table 3.

CONCLUSION

We have presented the derivation of higher order Newton Cotes closed formulae (orders p = 8, 9, 10, 11) by the conventional derivation technique of constant polynomials and the derivation using the Lagrange's interpolation functions for orders p = 1, 2,..., 11. It is obvious in this report that the results coincide with that of the conventional techniques. Whereas, it is very complicated to use the conventional approach to derive higher orders, it is much easier to use the Lagrange's polynomial functions.

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