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The Effect of an Attached Mass on an Euler-Bernoulli Beam

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Abstract: The effect of an attached mass on an Euler-Bernoulli beam was considered to determine the frequency and the deflection of the beam. The governing equation was solved by analytical cum numerical method using separation of variables and mathematical program subject to the boundary conditions to arrived at a transcendental equation to obtain the roots of the equation. The case of 3 modes were considered and shown graphically, it was also discovered that the natural frequency reduced when the mass is attached and increased when the beam is free.

Key words: Attched mass, Euler-bernoulli, frequency, numerical method, deflection of beam

INTRODUCTION

The beam is one of the fundamental elements of engineering structures which is used in various structural application and machine designs which has been of interest to many researchers. Timoshenko (1992) worked on practical analytical techniques to determine the response of the beams with various boundary conditions. Steel (1967) worked on a finite beam with a moving load and discovered that load has damping effect on the beam. Stannistic and Hardin (1969) present a solution for simple supported beam which was interesting but not very easy to apply to different boundary conditions. Recently Scott Whitney (1999) studies the vibrations of cantilever beams and determines the frequency and the deflection of the beam; in the aforementioned they only considered the classical ends. In this study the non-classical ends and effect of an attached mass is considered which an extension is of the work of Scott Whitney.

DEFLECTION OF THE BEAM

Assume that the free end of a Bernoulli beam is subjected to a point load the beam will deflect into a curve. If the load is large the deflection y(x) will be greater. Suppose that the beam is subjected to a small deflection that is it is linearly elastic region and has a uniform cross section by Gere and Timoshenko (1997). The following differential equation of the motion of the beam can be integrated analytical or solved numerically to obtain the deflection. Thus the deflection is expressed due to the bending moment as

$$\frac{d^2Y}{d^2x} = \frac{-M}{EI} = k$$

This equation is called the curvature where EI is the flexural rigidity, E is the modulus of elastic of the beam and I is the moment of inertia.

The bending moment can also be related to the shear for v and the lateral load P on the beam, therefore

$$M = EI\frac{d^2Y}{d^2x} \tag{1a}$$

$$v = \frac{d^3Y}{dx^3} \tag{1b}$$

$$P = -EI\frac{d^4Y}{dx^4} \tag{1c}$$

For a particular load the distributed load, shear force and bending moment are given as:

$$P(x) = 0,\ v(x) = P,\quad M(x) = -PL\left(1 - \frac{x}{L}\right) \tag{2a}$$

Thus (1a) can be written

$$\frac{dY}{dx} = \int_{x=0}^{L} M(x) = -\frac{PL}{EI}\int_{x=0}^{L}\left(x - \frac{x^2}{2L}\right) \tag{2b}$$

Hence

$$Y(x) = \int_{x=0}^{L}\frac{dY}{dx}dx = -\frac{PL}{EI}\int_{x=0}^{L}\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) \tag{2c}$$

At free end of the beam the displacement is

$$Y(L) = -\frac{PL^3}{3EI} \tag{2d}$$

THE VIBRATIONAL EQUATION OF THE BEAM

The governing equation of the beam is given by the differential equation

$$EI\frac{d^4Y(x,t)}{dx^4} + m\frac{d^2Y(x,t)}{dt^2} = P(xt) \tag{3}$$

Subject to the following boundary conditions

$$Y(0,t)=0,\ Y'(0,t)=0,\ \ Y'''(L,t), = -\omega^2 m_1 Y'(L,t),\ \ Y''(L,t) = \omega^2 j_1 Y(L,t) \tag{4}$$

When a force is removed from a displaced beam, the beam will return to us original shape. However, the beam will vibrate around the initial location due to the vertical of the beam. Suppose the inertia, the modulus of elasticity and the cross-sectional area A are constant along the length of the beam the equation of vibration now becomes

$$EI\frac{\partial^4 Y}{\partial x^4} = -m\frac{\partial^2 Y}{\partial t^2} \tag{5}$$

Where m is the mass per unit length. This (Atkins, 1994) can be solved analytically by method of separation of variables by assume a solution of the form

$$Y(X,t) = X(x\ \ T(t)) \tag{6}$$

By substituting Eq. 6 into 5, two differences equating were obtained and they are written as

$$\frac{\partial^4 X(x)}{\partial x^4} - \beta_n^4 X(x) = 0 \tag{7a}$$

$$\frac{\partial^2 T(t)}{\partial t^2} + W_n^2\ T(t) = 0 \tag{7b}$$

$$\text{where}\ \ \beta_n^4 = \frac{W_n^2 m}{EI} \tag{7c}$$

Subject to the boundary conditions and the initial conductions,

$$x(0) = x'(0) = 0,\ \ x'''(L) = -w^2 m_1\ x(L),\ x''(L) = w^2 j_1\ x(L) \tag{7d}$$

$$T(0) = \dot{T}(0) = 0$$

On solving Eq. 5a gives a solution as a linear combination of trigonometric equations by Volt era and Zachmangolou (1965).

$$X(x) = a_1[\cos(\beta_n x) + \cosh(\beta_n x)] + a_2\left[\cos(\beta_n x) - \cosh(\beta_n x)\right] + a_3\left[\sin(\beta_n x) + \sin(\beta_n x)\right] + a_4\left[\sin(\beta_n x) - \sinh(\beta_n x)\right] \tag{8}$$

Substitute (7c) in (8) reduce the equation to

$$X_n(x) = 2C_2 \begin{bmatrix} \left(1 + \cos(\beta_n L)\cosh(\beta_n L) = \frac{j_1\beta_n 3}{m} + \frac{m_1\beta_n}{m}\right) \\ \sin(\beta_n L)\cosh(\beta_n L) + \\ \left(\frac{m_{1_n}\beta_n}{m} - \frac{j_1\beta_n 3}{m}\right)\cos(\beta_n L)\sin(\beta_n L) + \frac{\beta_n 4 j_1 m_1}{m^2} \\ \left(1 - \cos(\beta_n L)\cosh(\beta_n L)\right) \end{bmatrix} = 0 \tag{9}$$

Since $2C_2$ is an arbitrary value and cannot be zero. Hence (9) can be written as

$$1 + \cos(\beta_n L)\cosh(\beta_n L) - \left(\frac{j_1(\beta_n L)^3}{mL^3} + \frac{m_1(\beta_n L)}{mL}\right) \sin(\beta_n L)\cosh(\beta_n L) + \left(\frac{m_1(\beta_n L)}{mL} - \frac{j_1(\beta_n L)^3}{mL^3}\right) \cos(\beta_n L)\sin(\beta_n L) + \frac{m_1 j_1(\beta_n L)^4}{m^2 L^4}\left(1 - \cos(\beta_n L)\cosh(\beta_n L)\right) = 0 \tag{10}$$

Equation 10 is called the transcendental equation which is then solved to obtain the roots of the equation. Solve Eq. 7b to obtain.

$$Y_n(x,t) = X_n(x)\left[A_n \cos W_n t + \beta_n \sin(W_n t)\right] \tag{11}$$

A_n depends on the initial position at time t = 0 and β_n depends on initial velocity. In this case it is assumed that the beam starts its vibration when displaced at rest thus $\beta_n = 0$, therefore

$$A_n = \frac{2}{L}\int_0^L v(x,t=0)X_n(x)\partial x \tag{12}$$

By substituting the initial displacement obtained earlier into Eq. 12 A_n can be written as

$$A_n = \frac{-2P}{3L\beta_n^4 EI\left[\begin{array}{l}\left(2\sin(\beta_n L)e^{\beta_n L} + e^{2\beta_n L} - 1\right) + \\ G\left(2\cos(\beta_n L)e^{\beta_n L} - e^{2\beta_n L} - 1\right)\end{array}\right]} *$$

$$\left[\begin{array}{l}\sin(\beta_n L)\left[\begin{array}{l}3\left(e^{2\beta_n L} - 1\right) + G(\beta_n L)^2\left(e^{2\beta_n L} - 1\right) - \\ G(\beta_n L)^3\left(e^{2\beta_n L} + 1\right)\end{array}\right] \\ + \cos(\beta_n L)\left[\begin{array}{l}3 - 3e^{2\beta_n L} + G(\beta_n L)^3\left(e^{2\beta_n L} + 1\right) + \\ 3G\left(e^{2\beta_n L} + 1\right) + \frac{G(\beta_n L)^3}{2}\left(e^{2\beta_n L} - 1\right)\end{array}\right] \\ -2(\beta_n L)^3 e^{\beta_n L} - 3Ge^{\beta_n L}\end{array}\right] \quad (13)$$

Where $G = \frac{j_1 \beta_n^3}{m}$

By assigning the numerical values as

$j_1 = 0.5$, $m = 7.04$, $EI = 215280$, $P = 1000$ N, $L = 10$ m

Two cases were considered and were shown from the Table 1.

Case I: The root of the equation, the values of A_n and the natural frequency when the load was attached with mass were obtain in the form of the Table 1.

Case II: The root of the equation, the values of A_n and the natural frequency were obtain in the form of the Table 2 when there is no attached mass that is when $j_1 = 0$ and $m_1 = 0$.

Table 1: The frequency of the beam increases when the mass is not attached

N	$\beta_n L$	A_n (m)	ω_n (Hz)
1	1.071617603	-2.10041294*10^{1}	2.025249665
2	3.981842254	-1.757877318*10^{5}	27.72576989
3	7.099751359	-5.284033829*10^{8}	88.14583412
4	10.22676265	-8.15088089*10^{11}	182.8908027
5	13.35565353	-9.50469751*10^{14}	311.9218522
6	16.48297035	-9.28558467*10^{17}	475.1015722
7	19.60690008	-8.08488771*10^{20}	672.253933
8	22.72691656	-4.75661909*10^{23}	903.1547085
9	25.83891313	-6.44227633*10^{26}	1165.519106
10	28.94402446	-3.35926696*10^{29}	1464.985566
11	32.03951117	-2.17373828*10^{32}	1795.094332

Table 2: The frequency of the beam decreases when the mass is attached

N	$\beta_n L$	A_n (m)	ω_n (Hz)
1	1.875104069	0.01592730682	329.70
2	4.694091133	-1.0088990438	2066.20
3	7.854757438	0.00019177692	5785.50
4	10.99554073	-0.00006998826	11337.20
5	14.1371688839	0.00003288057	18741.30
6	17.2787599953	-0.0000180088996	27996.20
7	20.4203552251	0.00001091030	39101.61
8	23.5619449001	-0.0000071100022	52058.37
9	26.70353775602	0.0000044887779	66866.07
10	29.8451302105	-0.000003349467	83524.74

RESULTS AND DISCUSSION

Typical values of A_n and the frequency for the beams are shown in Table 1 and 2, it was discovered from Table 1 that the frequency of the beam increases when the mass is not attached and from Table 2 the frequency of the beam decreases when the mass is attached, also Fig. 1-3 show the deflection of the beam caused by each modes when the mass was attached. As the time progresses, each modes vibrate around the zero deflection

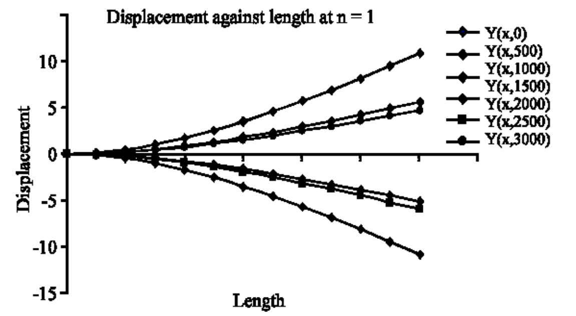

Fig. 1: Vibration of 1st characteristics mode

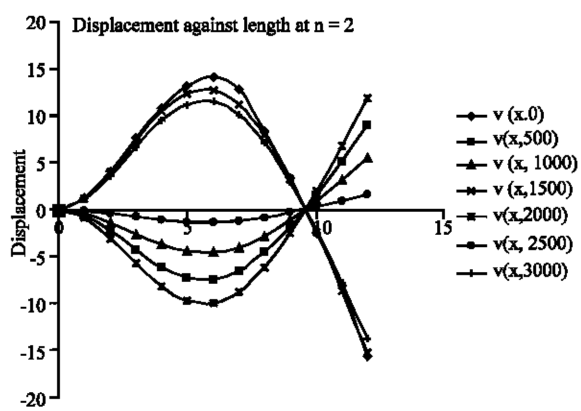

Fig. 2: Vibration of second characteristics mode

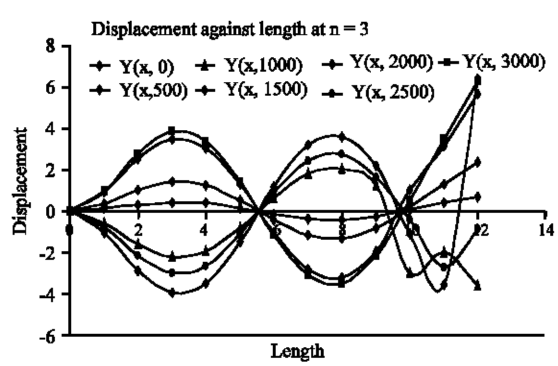

Fig. 3: Vibrations of third characteristic mode

line with the frequency listed in Table 1 and 2, Fig. 1, 2 and 3 shows the vibration for the first three modes the higher modes act similarly.

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