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Coefficient Inequalities and Convolution Properties Associated with Certain Classes of Analytic Functions

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Abstract: In the present study we defined two classes of analytic functions with negative coefficients using Salagean derivative operator. Coefficient inequalities and convolution properties associated with these classes are investigated

Key words: Coefficient inegualities, convolution properties, certain classes, analytic function

INTRODUCTION

Let $T(\rho)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=\rho}^{\infty} a_\rho z^\rho,\ (a_k \geq 0,\ \rho \in N \setminus \{1\} = \{2,3,\ldots\}) \quad (1)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$

A function $f(z) \in T(\rho)$ is said to be in the class $T_{n,\rho}(\lambda,\beta)$ if it satisfies the condition

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1-\lambda)D^n f(z)}\right\} > \beta \quad (2)$$

For some $0 \leq \beta < 1,\ 0 \leq \lambda < 1,\ n = 0,1,2,\ldots,\ z \in E$ and D is the same as Salagean derivative operator defined as

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z), \ldots$$
$$D^n f(z) = z(D^{n-1}f(z))' \text{ (Abdulhalim, 1992)} \quad (3)$$

Also, let $C_{n,\rho}(\lambda,\beta)$ denote the subclass of $T(\rho)$ of all functions f(z) satisfying the following inequality

$$\mathrm{Re}\left\{\frac{D^{n+2}f(z)}{(1-\lambda)D^{n+1}f(z) + \lambda D^{n+2}f(z)}\right\} > \beta \quad (4)$$

for some $0 \leq \beta < 1, 0 \leq \lambda < 1, n = 0,1,2,\ldots,\ z \in E$ and D is the same as defined in (3).

In particular

$$C_{0,2}(\lambda,\beta) \equiv C_2(\lambda,\beta) \text{ and } T_{o,2}(\lambda,\beta) \equiv T_2(\lambda,\beta)$$

were studied by Altintas and Owa (1988). Also, putting $n = 0, \lambda = 0$ we obtain the classes $T_\rho(0,\beta) \equiv T_\rho(\beta)$ and $C_\rho(0,\beta) \equiv C_\rho(\beta)$ which were investigated by Choi and Kim (1996). They are subclasses of order β and convex of order β, respectively see (Srivastava and Owa, 1992; Duren, 1983)

Let $f_j(z) \in T(\rho)$ $(j = 1,2,\ldots,m)$ be given by

$$f_j(z) = z - \sum_{k=\rho}^{\infty} a_{k,j} z^k \quad (5)$$

then the convolution (or Hadamard product) $f_j(z)$ of is defined by

$$\prod_{j=1}^{m} f_j(z) = (f_1 * \ldots * f_m)(z) = z - \sum_{k=p}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}\right) z^k \quad (6)$$

Owo and Srivastava (2003), Saitoh and Owa (2001).

Here we define $D^n f(z)$ by

$$D^n f(z) = z - \sum_{k=\rho}^{\infty} k^n a_k z^k \quad (7)$$

COEFFICIENT INEQUALITIES

Lemma 1: A function f(z) defined by (1) is in $T_{n,\rho}(\lambda,\beta)$ if and only if

$$\sum \left[k^{n+1} - \beta(\lambda k^n (k-1) + k^n)\right] a_k \leq 1 - \beta \quad (8)$$

for some $0 \leq \beta < 1, 0 \leq \lambda < 1, n = 0,1,2,\ldots,$.

Proof: Since $f(z) \in T(\rho)$. Then by using (7) in (2) with some simple transformation the result follows.

Lemma 2: A function f(z) defined by (1) is in $C_{n,\rho}(\lambda,\beta)$ if and only if

$$\sum\left[k^{n+2} - \beta(\lambda k^{n+1}(k-1) + k^{n+1})\right]a_k \le 1-\beta \qquad (9)$$

for some $0 \le \beta < 1, 0 \le \lambda < 1, n = 0,1,2,...,$

Proof: Since $f(z) \in T(\rho)$. Then, using (7) in (4) with some simple transformation the result follows.

Lemma 1 and 2 shall be used in obtaining our next results.

CONVOLUTION PROPERTIES

Theorem 1: If $f_j(z) \in T_{n,\rho}(\lambda,\beta), (j = 1,2,...,m)$ then

$$\prod_{j=1}^{m} f_j(z) \in T_{n_j,\rho}(\lambda,\gamma_m)$$

Where

$$\gamma_m = 1 - \frac{(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m} 2^{\sum_{j=1}^{m} n_j - n}(2-(1-\lambda)\beta_j) - (1-\lambda)\prod_{j=1}^{m}(1-\beta_j)} \qquad (10)$$

$$\text{and } \sum_{j=1}^{m} n_j - n \ge 0$$

Proof: We need to find the largest γ_m such that

$$\sum_{k=\rho}^{\infty}\left[k^{n+1} - \gamma_m(\lambda k^n(k-1) + k^n)\right]\prod_{j=1}^{m}(a_{k,j}) \le 1-\gamma_m \qquad (11)$$

Note that if m = 1, then $\gamma_1 = \beta_1$. Now suppose m = 2. Then for functions

$f_1(z) \in T_{n_1,\rho}(\lambda,\beta_1)$, and $f_2(z) \in T_{n_2,\rho}(\lambda,\beta_2)$ wehave

$$\sum_{k=\rho}^{\infty}\left[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})\right]a_{k,1} \le 1-\beta_1$$

and

$$\sum_{k=\rho}^{\infty}\left[k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})\right]a_{k,2} \le 1-\beta_2$$

so that

$$\sum_{k=\rho}^{\infty}\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})]}{1-\beta_1}a_{k,1} \le 1$$

and

$$\sum_{k=\rho}^{\infty}\frac{[k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})]}{1-\beta_2}a_{k,2} \le 1.$$

Hence by Cauchy-Schwarz inequality we have

$$\sum_{k=\rho}^{\infty}\sqrt{\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})][k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})]}{(1-\beta_1)(1-\beta_2)}(a_{k,1})(a_{k,2})} \le 1 \qquad (12)$$

In order to prove that $(f_1 * f_2)(z) \in T_{n,\rho}(\lambda,\gamma_2)$ it is sufficient to show that

$$\sqrt{(a_{k,1})(a_{k,2})} \le \frac{1-\gamma_2}{k^{n+1} - \gamma_2(\lambda k^n(k-1) + k^n)}\sqrt{\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})][k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2})]}{(1-\beta_1)(1-\beta_2)}}$$

since

$$\sum_{k=\rho}^{\infty}[k^{n+1} - \gamma_2(\lambda k^n(k-1) + k^n)](a_{k,1})(a_{k,2}) \le$$

$$\le 1-\gamma_2\sum_{k=\rho}^{\infty}\sqrt{\frac{[k^{n_1+1} - \beta_1(\lambda k^{n_1}(k-1) + k^{n_1})][k^{n_2+1} - \beta_2(\lambda k^{n_2}(k-1) + k^{n_2}}{(1-\beta_1)(1-\beta_2)}(a_{k,1})(a_{k,2})}$$

$$\leq 1-\gamma_2\left[\left(\frac{\sum_{k=\rho}^{\infty}[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})]}{1-\beta_1}(a_{k,1})\right)\left(\frac{\sum_{k=\rho}^{\infty}[k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{n_2})]}{1-\beta_2}(a_{k,2})\right)\right]^{\frac{1}{2}}$$

$$\leq 1-\gamma_2$$

But from 12 we have for all $k=\rho=2,3,...$

$$\sqrt{(a_{k,1})(a_{k,2})}\leq \sqrt{\frac{(1-\beta_1)(1-\beta_2)}{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}(k-1)+k^{n_2})]}}$$

Hence it is sufficient to find the largest γ_2 such that

$$\sqrt{\frac{(1-\beta_2)(1-\beta_2)}{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{n_2})]}}\leq$$

$$\leq\frac{1-\gamma_2}{[k^{n+1}-\gamma_2(\lambda k^{n}(k-1)+k^{n})}\sqrt{\frac{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{2})]}{(1-\beta_1)(1-\beta_2)}}$$

That is,

$$1\leq\frac{1-\gamma_2}{k^{n+1}-\gamma_2(\lambda k^{n}(k-1)+k^{n})}\left[\frac{[k^{n_1+1}-\beta_1(\lambda k^{n_1}(k-1)+k^{n_1})][k^{n_2+1}-\beta_2(\lambda k^{n_2}(k-1)+k^{n_2})]}{(1-\beta_1)(1-\beta_2)}\right]$$

$$\gamma_2\leq 1+\frac{(\lambda(k-1)+1-k)\prod_{j=1}^{2}(1-\beta_j)}{\prod_{j=1}^{2}k^{\sum_{j=1}^{2}n_j-n}\left(k-\beta_j(\lambda(k-1)+1)\right)-(1+\lambda)\prod_{j=1}^{2}(1-\beta_j)} \quad (13)$$

$$\gamma_2=1-\frac{(1-\lambda)\prod_{j=1}^{2}(1-\beta_j)}{\prod_{j=1}^{2}2^{\sum_{j=1}^{2}n_j-n}\left(2-(1+\lambda)\beta_j\right)-(1+\lambda)\prod_{j=1}^{2}(1-\beta_j)}$$

Now suppose that

$$\prod_{j=1}^{m}{}^{*}f_j(z)\in T_{n,\rho}(\lambda,\gamma_m)$$

Where

$$\gamma_m=1-\frac{(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}2^{\sum_{j=1}^{m}n_j-n}\left(2-(1+\lambda)\beta_j\right)-(1+\lambda)\prod_{j=1}^{m}(1-\beta_j)}$$

Then repeating the process above we obtain that

$$\prod_{j=1}^{m+1}{}^{*}f_j(z)\in T_{n,\rho}(\lambda,\gamma_{m+1})$$

and

$$\gamma_{m+1}=1-\frac{(1-\lambda)\prod_{j=1}^{m+1}(1-\beta_j)}{\prod_{j=1}^{m+1}2^{\sum_{j=1}^{m+1}n_j-n}\left(2-(1+\lambda)\beta_j\right)-(1+\lambda)\prod_{j=1}^{m+1}(1-\beta_j)}.$$

Hence the conclusion follow by induction.

Theorem 2: If $f_j(z)\in C_{n,\rho}(\lambda,\beta),\ (j=1,2,...,m)$ then

$$\prod_{j=1}^{m}f_j(z)\in C_{n_j,\rho}(\lambda,\gamma_m)$$

Where

$$\gamma_m=1-\frac{(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}2^{\sum_{j=1}^{m}n_j-n+1}(2-(1-\lambda)\beta_j)-(1-\lambda)\prod_{j=1}^{m}(1-\beta_j)}$$

$$\text{and } \sum_{j=1}^{m} n_j - n + 1 \geq 0$$

Proof: The proof is similar to that of Theorem 1 by using Lemma 2.

Remarks: Putting $\lambda = 0$ and $n = n_j = 0$ in our results we obtain the results in Choi and Kim (1996).

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