Journal of Engineering and Applied Sciences 2 (5): 874-876, 2007

On Word Equation of the Form $z_1^{k_1} z_2^{k_2} z_3^{k_3} \ldots z_n^{k_n} = \chi^k$ In a Free Semigroup: A Further Extension

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Abstract: Word equations of the form $z_1^{k_1} z_2^{k_2} z_3^{k_3} \ldots z_n^{k_n} = x^k$ are concerned in this study. In particular, I investigate the case where x is of different length than Z_i, for any i and k and k_i are at least 3, for all powers of the same word for all $1 \le i \le n$. It is also shown that this result implies a well-known result by Appel and Djorup about the more special case where $k_i = k_j$ for all $1 \le i < j \le n$.

Key words: Word equations, Appel and Djorup's result

INTRODUCTION

Word equation of the form

$$z_1^{k_1} z_2^{k_2} z_3^{k_3} \ldots z_n^{k_n} = x^k \qquad (1)$$

have long been interest,(Appel and Djorup. 1998; Lentin, 1965; Lyndon and Schutzenberger, 1962). Originally motivated from questions concerning equations in free groups of special cases of in free semi-groups were investigated. For example:

$$z_1^{k_1} z_2^{k_2} = x^k$$

is of rank 1 which was shown by Lyndon and Schutzen berger (1962) and Lentin (1965) investigated the solutions of:

$$z_1^{k_1} z_2^{k_2} z_3^{k_3} = x^k$$

which has solutions of higher rank, see Example 1 and Appel and Djorup (1998) investigated

$$z_1^{k} z_2^{k} z_3^{k} \ldots z_n^{k} = x^k$$

We show in theorem 3 of this study that equations of the form (1) are rank of 1, if all exponents are larger than 2 and $n \le k$ and x is not a conjugate of z_i for any $1 \le i \le n$. This result straightforwardly implies theorem 4 by Appel and Djorup (1998).

We continue with fixing some notations. More basic definition can be found in (Lothaire, 1983). Let A be a finite set and let A^* be a free monoid generated by A. We call A alphabet and the elements of A^* words. Let $w = w_{(1)} w_{(2)} \ldots w_{(n)}$, where $w_{(i)}$ is a letter, for every $1 \le i \le n$. We denote the length n of w by |w|. An integer $1 \le p \le n$ is a period of w, if $w_{(i)} = w_{(i+p)}$ for all $1 \le i \le n-p$. A non-empty word u is called a border of a word w, if w = uv = v`u for some suitable words v and v`. We call w bordered, if it has a border that is shorter than w, otherwise w is called unbordered. A word w is called primitive if $w = u^k$ implies that k = 1. We call two words u and v conjugates, denoted by u~v, if u = xy and v = yx for some words x and y. Let $[u] = \{v \mid u \sim v\}$ and $w^* = \{w_i \mid i \ge 0\}$.

Let $\sum$ be an alphabet. A tuple $(u, v) \in \sum^* \times \sum^*$ is called word equation in $\sum$, usually denoted by u = v. Let $u, v \in \sum^*$ be such that every letter of $\sum$ occurs in u or v.

A morphism denoted by $\varphi: \sum^* \rightarrow A^*$ is called solution of u = v, if $\varphi(u) = \varphi(v)$. The rank of a solution φ of an equation u = v is the maximum rank of a free subsemigroup that contains $\varphi(\sum)$. The rank of an equation is the maximum rank of all its solutions.

RESULTS

The following theorem was shown by Fine and WIf, (1965). As usual gcd denotes the greatest common divisor.

Theorem 1: Let $w \in A^*$ and p and q be periods of w. If $|w| \ge p+q-\gcd\{p,q\}$ then $\gcd\{p,q\}$ is a period of w.

The following lemma is consequence of theorem 1; (Halava *et al.*, 2000).

Lemma 1: Let $w \in A^*$ and p be the smallest period of w. Then for any period q of w, with $q \le |w| - p$, we have that q is a multiple of p.

The following theorem follows Lyndon and Schutzenberger (1962) 's proof for free groups. Harju and Nowtha (2004) presents a short direct proof and the following Lemma 2.

Theorem 2: Let x, y, $z \in A^*$ and i, j, $k \geq 2$. If $x^i = y^j z^k$ then x, y, $x \in w^*$ for some $w \in A^*$.

Lemma 2: Let x, $z \in A^*$ be primitive and non-empty words. If z^m is a factor of x^k for some k, $m \geq 2$, then either $(m-1)|z| < |x|$ or z and x are conjugates.

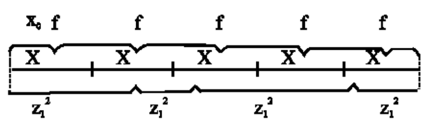
Proof: Assume that $(m-1)|z| \geq |x|$. Then z^m has two periods $|x|$ and $|z|$ and hence, a period $\gcd(|x|, |z|)$ by theorem 1. Now, $|x| = |z|$ and x and z are conjugates.

The following theorem is the main result of this study. It shows that the solutions of a word equation of the form $z_1^{k_1} z_2^{k_2} z_3^{k_3} \ldots z_n^{k_n} = x^k$ are necessarily of rank 1 under certain conditions.

Theorem 3: Let $n \geq 2$ and x, $z_i \in A^*$ and $|x| \neq |z_i|$ and k, $k_i \geq 3$, for all $1 \leq i \leq n$. If $z_1^{k_1} z_2^{k_2} z_3^{k_3} \ldots z_n^{k_n} = x^k$ and $n \leq k$ then x, $z_i \in w^*$, for some $w \in A^*$ and all $1 \leq i \leq n$.

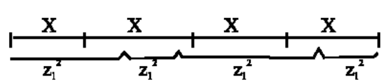
Proof: Assume that x, z_i for all $1 \leq i \leq n$, are primitive words. Note that $|z_i^{k_i - 1}| < |x|$ by lemma 2 and therefore $|z_i| < |x|$ for all i.

If n < k then let f be an unbordered conjugate of x and $x^k = x_0 f^{k-1} x^1$ with $x = x_0 x_1$. Let us illustrate this case with the following drawing.



By the pigeon hole principle there exists an i such that f is factor of $z_i^{k_i}$ But now, f is bordered; a contradiction.

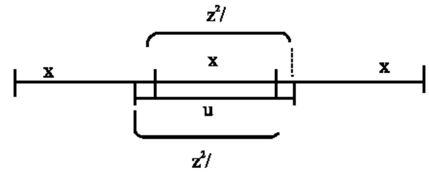
Assume that, n = k in the following. Let us illustrate this with the following drawing:



From $k_i \geq 3$, for all $1 \leq i \leq n$, follows that there exits a primitive word $z \in A^*$ such that for every i with $|x| \leq |z_i^{k_i}|$ we have that $|z_i|$ is the smallest period of x and $z_i \in [z]$ by lemma 1.

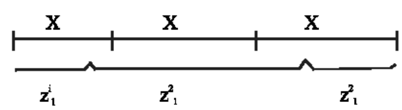
There exits an i such that $|x| \leq |z_i^{k_i}|$ by a length argument. We also have for all $1 \leq i \leq n$ that, if $|x| \leq |z_i^{k_i}|$ then $|z_{i+1}^{k_{i+1}}| < |x|$, otherwise either z is not primitive or $x \in Z_0^*$, with $z_0 \in [z]$ and x is not primitive. Similarly for z_{i-1}. Moreover, we have that all factors $z_j^{k_j}$ with $|x| \leq |z_j^{k_j}|$ occur in a word u which is a factor of xxx and $|u| < |x| + |z|$ otherwise z^{k_i+1}, for some $1 \leq i \leq n$ and xx have a common factor of length greater or equal to $|x| + |z|$ and either x or z is not primitive.

Consider the following drawing:



Therefore, we have for every i with $|x| \leq |z_i^{k_i}|$ that $|z_{i+1}^{k_{i+1}}| < |zz|$ because $|z_{i+1}| < z$ and otherwise z is not primitive. This proves the case for n >3 since then $|z_i^{k_i} z_{i+1}^{k_{i+1}}| < |xx|$, for every i such that $|x| \leq |z_i^{k_i}|$ and $|z_1^{k_1} z_2^{k_2} z_3^{k_3} \ldots z_n^{k_n}| < |x^k|$; a contradiction.

The case k = 3 remains. Since we can construct from one equation a new one of the same rank by cyclic shifts, we can assume that $|x| \leq |z_2^{k_2}|$. Let us consider the following drawing:



By the arguments above, we have that $|z_1^{k_1}| < |x|$ and $|| < |x|$. Now, $|z^{k_2 - 1}| < |x| < |z^{k_2}|$ and $|z^{k_2}| < |z_1^{k_1}| + |z_3^{k_3}|$. Let $x = z'^{k_2 - 1} z'_0$, where $z'_0 \in [z]$ and z' is a prefix of z'. Let g be an unbordered conjugate of such that $z'z' = g_1 g g_0$, where $g = g_0 g_1$ and $z' = g_1 g_0$. We get a contradiction, if $|g_1 g| \leq |z_1^{k_1}|$ since then $z_1^{k_1}$ covers g and hence, g is bordered. So, assume $|g_1 g| > |z_3^{k_3}|$. But now, $|z_1^{k_1} z_2^{k_2}| < |xxg_1|$, since $|g_0 z'_0 x| < |z_2^{k_2}| < |x| + |z| < |g_0 z'_0 x g_1|$ and g is covered by $z_3^{k_3}$; a contradiction again.

The following example shows why the condition $|x| \neq |z_i|$ is needed in theorem 3.

Example 1: Consider $x^4 = z_1^1 z_2^2 z_3^3$. There exits a solution φ of rank 2 with $\varphi(x) = \varphi(z^1) = a^3 b^3$ and $\varphi(z^3) = b^3$.

Theorem 3 implies the following result by Appel and Djorup (1998).

Theorem 4: Let $n \geq 2$ and $x, z_i \in A^*$, for all $1 \leq i \leq n$. If $z_1^{k_1} z_2^{k_2} z_3^{k_3} \ldots z_n^{k_n} = x^k$ with $n \leq k$, then $x, z_i \in w^*$, for some $w \in A^*$ and all $1 \leq i \leq n$.

Proof: If n = 2 the result follows from theorem 2. Assume n >2 in the following. Let $\overline{x}$ and $\overline{z}_i$ denote the primitive roots of $x = \overline{x}^{\ell}$ and $z_i = \overline{z}_i^{\ell_i}$, for all $1 \leq i \leq n$, respectively. Then we have:

$$\overline{z}_1^{\ell_1 k} \overline{z}_2^{\ell_2 k} \ldots \overline{z}_n^{\ell_n k} = \overline{x}^{\ell k} \qquad (2)$$

If there exists an i such that $|\overline{z}_i| = |\overline{x}|$ then $\overline{z}_i \sim \overline{x}$ and we have the equation:

$$\overline{z}_1^{\ell_1 k} \overline{z}_2^{\ell_2 k} \ldots \overline{z}_{i-1}^{\ell_{i-1} k} \overline{z}_{i+1}^{\ell_{i+1} k} \ldots \overline{z}_n^{\ell_n k} = \overline{x}^{(\ell - \ell_i) k} \qquad (3)$$

Which has not a higher rank than (2). Since (3) meets our assumptions this reduction can be iterated until either $n = 2$ or $|\overline{z}_i| \neq |\overline{x}|$ for all $1 \leq i \leq n$. But, then theorem 3 gives the result.

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