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Generalized Coefficient Inequalities for Certain Multivalent and Meromorphically Multivalent Analytic Functions

Oladipo Abiodun Tinuoye
Department of Pure and Applied Mathematics, University of Technology,
Ladoke Akintola, P.M.B. 4000, Ogbomoso, Nigeria

Abstract: In this study we prove some theorems involving certain generalized coefficient inequalities for multivalent and meromorphically multivalent functions defined by Salagean differential operator.

Key words: Coefficient, multivalent, theorms, inqualities

INTRODUCTION

Let T (p) and M (p) denote the classes of functions f (z) and g (z) of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k \quad (p\in N=\{1,2,...\}) \tag{1}$$

and

$$g(z)=z^{-p}+\sum_{k=p}^{\infty}a_k z^k \quad (p\in N) \tag{2}$$

which are analytic and multivalent in the unit disk $E=\{z : |z|<1$ and in the punctured unit disk $U=\{z : |z|<1,$ respectively (Irmak and Owa, 2003).

A function $f \in T(p)$ is said to be multivalent starlike in E if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \quad z\in E \tag{3}$$

and multivalent convex if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \quad z\in E \tag{4}$$

Furthermore, it is multivalent close-to-convex if it satisfies

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>0 \quad z\in E \tag{5}$$

Also, a function $g(z) \in M(p)$ is said to be meromorphically multivalent starlike in U if it satisfies the condition

$$\mathrm{Re}\left\{-\left(\frac{zg'(z)}{g(z)}\right)\right\}>0 \quad z\in U \tag{6}$$

and meromorphically multivalent convex if it satisfies the inequality

$$\mathrm{Re}\left\{-\left(1+\frac{zg''(z)}{g'(z)}\right)\right\}>0 \quad z\in U \tag{7}$$

see (Irmak and Owa, 2003) for details.

Here the author wish to give the following definitions of the subclasses to be considered in this study.

$$T_n(p,\alpha)=\{f\in T(p): \mathrm{Re}\frac{D^{n+1}f(z)}{D^n f(z)}>\alpha,\ 0\le\alpha<1,\ z\in E\} \tag{8}$$

$$K_n(p,\alpha)=\{f\in T(p): \mathrm{Re}\frac{D^n f(z)}{p^n z^p}>\alpha,\ 0\le\alpha<1,\ z\in E\} \tag{9}$$

$$T_n(p,\alpha)=\{f\in T(p): \mathrm{Re}\left(-\frac{D^{n+1}g(z)}{D^n g(z)}\right)>\alpha,\ 0\le\alpha<1,\ z\in U\} \tag{10}$$

where D^n is the Salagean differential operator defined as

$$D^0 f(z)=f(z),\ D^1 f(z)=zf'(z),\ D^n f(z)=z(D^{n-1}f(z))' \tag{11}$$

$n = 0, 1, 2....,$ $z \in (E, U)$ (Salagean, 1983).

Let $p \in P$ such that $P(z)$ is regular in E and satisfies the conditionsp(0) = 1 and Re $p(z) > 0$ in E.

The aim of the present study is to estimate coefficient bounds for each of the classes defined in (8) above. To do this we need the following lemma.

Lemma A. Aini *et al.* (2006). If $p \in P$ then $|C_k| \le 2$ for each k.

Lemma B. Ram (1973), Nehari and Netanyahu (1971). Let $h(z) = 1 + \beta_1 z + \beta_2 z^2 + ...$ and $1 + G_1(z) = 1 + b'_1 z + b'_2 z^2$ be the functions of the class P and set

$$\gamma_v = \frac{1}{2^v}\left\{1 + \frac{1}{2}\sum_{\mu=1}^{v}\binom{v}{\mu}\beta_\mu\right\}, \quad \gamma_0 = 1$$

If A_n is defined by

$$\sum_{v=1}^{\infty}(-1)^{v+1}\gamma_{v-1}G_1^v(z) = \sum_{v=1}^{\infty}A_v z^v$$

then
$|A_n| \leq$ Ram (1973), Nehari and Netanyahu (1971).

THEOREMS AND PROOFS

We state and proof the following

Theorem 1: Let $f \in T_n(p, a)$. Then we have the following inequalities

(i) $$|a_2| \leq \frac{(1-\alpha)-2^{n-1}}{2^{n-1}}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (12)$$

(ii) $$|a_3| \leq \frac{3(1-3^{n-1}) + \alpha(2\alpha-5)}{3^n}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (13)$$

(iii) $$|a_4| \leq \frac{2(1-\alpha)[(1-\alpha)(5-2\alpha)+(1-2^n)-3\cdot 4^n]}{3\cdot 4^n}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (14)$$

Proof: Since $f \in T_n(p, a)$,we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \alpha + (1-\alpha)p(z) \qquad (15)$$

for some $p(z) \in P$. Setting $p(z) = 1 + c_1 z + c_2 z^2 + ... + ... c_n z^n$ and comparing coefficients in (15) we obtain

$$2^n a_2 = (1-\alpha)c_1 - 2^n \qquad (16)$$

$$2\cdot 3^n a_3 = (1-\alpha)c_2 + 2^n c_1(1-\alpha)(1+a_2) - 2\cdot 3^n \qquad (17)$$

$$3\cdot 4^n a_4 = (1-\alpha)c_3 + 3^n c_1(1-\alpha)(1+a_3) + 2^n(1-\alpha)a_2 c_2 - 3\cdot 4^n \qquad (18)$$

Using the fact that

$$|c_k| \leq 2, \ k = p+1, \ p \in N = \{1,2,3,...\}$$

in (16), we at once obtain inequality (i). Eliminating a_2 a_3 as the case require , we obtain the inequalities in (12) and the Theorem is proved.

Theorem 2: Let $f \in K_n(p, a)$Then we have the following inequalities.

$$|a_2| \leq \frac{(1-\alpha)}{2^{n-1}}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (19)$$

$$|a_4| \leq \frac{2^{n+1}(1-\alpha)}{4^n}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (20)$$

$$|a_6| \leq \frac{2\cdot 3^n(1-\alpha)}{6^n}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (21)$$

$$|a_8| \leq \frac{2\cdot 4^n(1-\alpha)}{8^n}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (22)$$

$$|a_2 a_4 - a_3^2| \leq \frac{(1-\alpha)^2}{4^{n-1}}, \quad n = 0,1,2,..., 0 \leq \alpha < 1 \qquad (23)$$

Proof: Since $f \in K_n(p, a)$, we have

$$\frac{D^n f(z)}{p^n z^p} = \alpha + (1-\alpha)p(z), \qquad (24)$$

for some $p(z) \in P$. Taking p(z) as defined, and comparing the coefficients in (23) we obtain

$$2^n a_2 = (1-\alpha)c_1, \quad 0 \leq \alpha < 1, \ n = 0,1,2,... \qquad (25)$$

$$4^n a_4 = 2^n(1-\alpha)c_2 \qquad (26)$$

$$6^n a_6 = 3^n(1-\alpha)c_3 \qquad (27)$$

$$8^n a_8 = 4^n(1-\alpha)c_4 \qquad (28)$$

Following the same method in Theorem 1, the inequalities in Theorem 2 follow immediately.

We wish to inform here that all the coefficients of all odd powers of z are zeros. Thus the theorem is proved.

Additionally, we also note that the coefficient bounds of the functions in the subclass $M_n(p, a)$ being a punctured disk may not be obtainable.

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