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The General Equations of Motion in Prolate Spheroidal Coordinates and its Application to a Rain-Drop

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Abstract: In this study, the equations of motion for a prolate spheroidal massive body as well as its azimuthal solution has been formulated and applied to a physical system of rain-drop. It is shown that a prolate spheroidal (massive) body is stable (no abrupt change in its structure) as it transverse the spatial coordinate $(-\infty, \infty)$ but its geometry has its corresponding consequences and applications on the motion and structure of a particle in the gravitational field of such a body.

Key words: Prolate spheroidal, motion, rain-drop, implicit speed and azimuthal solution global warming

INTRODUCTION

Some five and a half decades of years ago, the theoretical study of gravitational fields was a matter almost exclusively treated to fields of massive bodies of perfectly spherical geometry (French, 1971) simply because of mathematical convenience and simplicity. An example is seen in the applications of Newton's Dynamical Theory of Universal Gravitation (NDTUG) in the treatment of the motion of particles (such as projectiles, satellite, penduli and even gas molecules) (Bowler, 1976) and the earth is treated under the general assumptions that the earth is a perfect sphere (Richtmyer, 1955). Similarly, in the solar system the motion of bodies (such as comets, planets, asteroids and stars) is treated entirely under the general assumption that the sun and these bodies are perfectly spherical in shape. In the same light, Einstein's Theory of Gravitation called General Relativity Theory (GRT), the motion of bodies (such as planets) and particles (such as photon) is treated under the assumption that the sun is exclusively a perfect sphere (the Schwarz child's space-time) (Moller, 1955). But the real fact of nature is that al rotating planets, stars and galaxies in the universe are either oblate or prolate spheroidal in shape.

It is known that satellites orbits around the earth are governed by NDTUG and the second harmonics (pole of order 3), as well as forth harmonics (pole of order 5) of gravitational scalar potential due to imperfect geometry. In 1952, Jeffrey's (1952) suggested the forth harmonics, which yielded amplitude of only 86% of the value obtained by King-Hele and Merson (1959) from the analysis of data on statistics orbits. In 1959, O'keefe, Eckels and Squires (1959) improved on Hele and Merson results using equatorial asymmetry for spherical shape. A year later, Vinti (1960) got a very good approximation of the second harmonics which reduced the problem of Sterne (1957) and Garfinkel (1958) quadratures by applying oblate spheroidal coordinates to investigate the motion of an earth satellite. Yet, there are still natural occurrences whose geometrical shapes are prolate spheroidal (such as rain drops) and their geometry will have corresponding consequences and effects in the motion of all particles in their gravitational fields as pointed out in references (Musongong and Howusu, 2005; Bakwa *et al*., 2003; Howusu and Musongong, 2005). It is worth mentioning that the literature of the previous work in this prolate spheroidal massive body has been carried out in (Musongong and Howusu, 2004) and Newton's gravitational potential for a homogeneous massive prolate spheroidal body formulated and solved, with the exact and complete results given in (2.25) and (2.26) for a field interior and exterior to the prolate spheroid. The above idea gives a motivation of the derivation of equations of motion and application to rain-drop and other physical situations in this study.

Mathematical analysis: Consider a massive homogeneous prolate spheroidal body as shown in the Fig. 1.

Let the Cartesian coordinates (x, y, z) be defined by the prolate spheroidal coordinates (η, ξ, ϕ) by (Arken, 1962; Hildebrand, 1962; Anderson, 1967) as:-

$$x = a\left[\left(\xi^2 - 1\right)\left(1 - \eta^2\right)\right]^{\frac{1}{2}} \cos\phi \qquad (1)$$

$$y = a\left[\left(\xi^2 - 1\right)\left(1 - \eta^2\right)\right]^{\frac{1}{2}} \sin\phi \qquad (2)$$

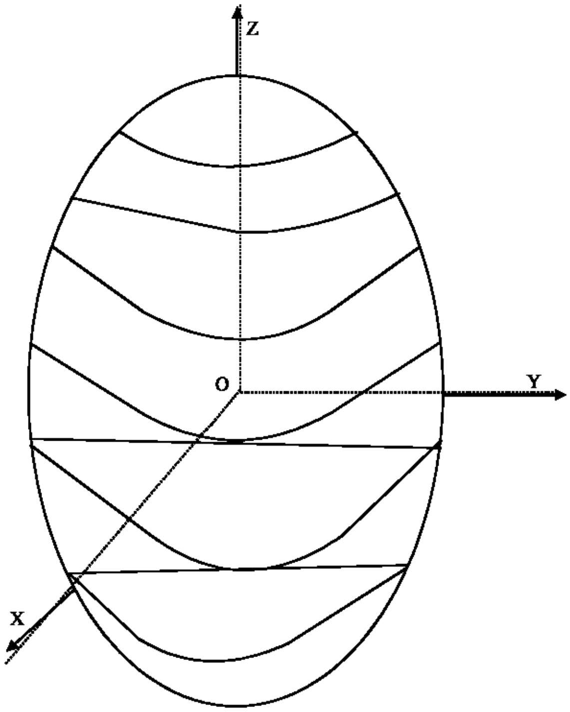

Fig. 1: A Homogeneous prolate spheroidal massive body

$$z = a\eta\xi \tag{3}$$

$$\{0 \le \xi < \infty, \qquad 0 \le \phi \le 2\pi\} \tag{4}$$

We chose η to be of same limit as ξ for mathematical convenience and physical applications.
Then the unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ are defined by the equations

$$\hat{i} = -\eta\left(\xi^2-1\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\cos\phi\hat{\eta} + \xi\left(1-\eta^2\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\cos\phi\hat{\xi} - \sin\phi\hat{\phi} \tag{5}$$

$$\hat{j} = -\eta\left(\xi^2-1\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\sin\phi\hat{\eta} + \xi\left(1-\eta^2\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\sin\phi\hat{\xi} + \cos\phi\hat{\phi} \tag{6}$$

$$\hat{k} = \xi\left(1-\eta^2\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\hat{\eta} + \eta\left(\xi^2-1\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\hat{\xi} \tag{7}$$

Conversely, the unit vectors in the prolate spheroidal coordinates are given by

$$\hat{\eta} = \left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\left[-\eta\left(\xi^2-1\right)^{\frac{1}{2}}\cos\phi\hat{i} - \eta\left(\xi^2-1\right)^{\frac{1}{2}}\sin\phi\hat{j} + \xi\left(1-\eta^2\right)^{\frac{1}{2}}\hat{k}\right] \tag{8}$$

$$\hat{\xi} = \left(\xi^2-\eta^2\right)^{-\frac{1}{2}}\left[\xi\left(1-\eta^2\right)^{\frac{1}{2}}\cos\phi\hat{i} + \xi\left(1-\eta^2\right)^{\frac{1}{2}}\sin\phi\hat{j} + \eta\left(\xi^2-1\right)^{\frac{1}{2}}\hat{k}\right] \tag{9}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j} \tag{10}$$

Consequently, the derivatives of the prolate unit vectors (where the dots represent time derivatives) are given by:-

$$\dot{\hat{\eta}} = \left(\xi^2-\eta^2\right)^{-1}\left\{\begin{matrix}\left[-\dot{\eta}\xi\left(\xi^2-1\right)^{\frac{1}{2}}\left(1-\eta^2\right)^{-\frac{1}{2}} - \dot{\xi}\eta\left(1-\eta^2\right)^{\frac{1}{2}}\left(\xi^2-1\right)^{-\frac{1}{2}}\right]\hat{\xi} \\ -\eta\left(\xi^2-1\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{\frac{1}{2}}\dot{\phi}\hat{\phi}\end{matrix}\right\} \tag{11}$$

$$\dot{\hat{\xi}} = \left(\xi^2-\eta^2\right)^{-1}\left\{\begin{matrix}\left[\dot{\xi}\eta\left(\xi^2-1\right)^{-\frac{1}{2}}\left(1-\eta^2\right)^{\frac{1}{2}} + \xi\dot{\eta}\left(1-\eta^2\right)^{-\frac{1}{2}}\left(\xi^2-1\right)^{\frac{1}{2}}\right]\hat{\eta} \\ +\xi\left(1-\eta^2\right)^{\frac{1}{2}}\left(\xi^2-\eta^2\right)^{\frac{1}{2}}\dot{\phi}\hat{\phi}\end{matrix}\right\} \tag{12}$$

$$\dot{\phi} = \left(\xi^2 - \eta^2\right)^{-\frac{1}{2}} \left[\eta\left(\xi^2 - 1\right)^{\frac{1}{2}} \hat{\eta} - \xi\left(1 - \eta^2\right)^{\frac{1}{2}} \hat{\xi} \right] \dot{\phi} \tag{13}$$

$$\vec{r} = a\left(\xi^2 - \eta^2\right)^{-\frac{1}{2}} \left[\xi\left(\xi^2 - 1\right)^{\frac{1}{2}} \hat{\xi} - \eta\left(1 - \eta^2\right)^{\frac{1}{2}} \hat{\eta} \right] \tag{14}$$

It follows after differentiating and re-arranging that the velocity in metric prolate spheroidal coordinates is given by

$$\frac{d\vec{r}}{dt} = a\left[\frac{\dot{\eta}\left(\xi^2 - \eta^2\right)^{\frac{1}{2}}}{\left(1 - \eta^2\right)^{\frac{1}{2}}} \hat{\eta} + \frac{\dot{\xi}\left(\xi^2 - \eta^2\right)^{\frac{1}{2}}}{\left(\xi^2 - 1\right)^{\frac{1}{2}}} \hat{\xi} + \dot{\phi}\left(1 - \eta^2\right)^{\frac{1}{2}} \left(\xi^2 - 1\right)^{\frac{1}{2}} \hat{\phi} \right] \tag{15}$$

From (Musongong and Howusu, 2004) for a particle exterior to the massive homogeneous prolate spheroid, the gravitational potential $\Phi(\eta, \xi)$ is given by the expression:-

$$\Phi^{+}_{(\eta,\xi)} = \frac{4\pi\rho_0 a^2 G}{3} \left\{ \frac{\xi_0^2 Q_0(\xi) P_0(\eta)}{\left\langle \frac{d}{d\xi} Q_0(\xi) \right\rangle_{\xi=\xi_0}} + \frac{Q_2(\xi) P_2(\eta) \left\langle \frac{d}{d\xi} P_2(\xi) \right\rangle_{\xi=\xi_0}}{3\left[Q_2(\xi) \left\langle \frac{d}{d\xi} P_2(\xi) \right\rangle_{\xi=\xi_0} - P_2(\xi) \left\langle \frac{d}{d\xi} Q_2(\xi) \right\rangle_{\xi=\xi_0} \right]} \right\} \tag{16}$$

The equations of motion in the gravitational field of the prolate spheroid can now be formulated as follows:-
Let the kinetic energy equation be given by

$$T(\eta, \xi, \phi) = \frac{1}{2} M(\eta, \xi, \phi) \left(\frac{d\vec{r}(\eta, \xi, \phi)}{dt} \right)^2 \tag{17}$$

But

$$\frac{d\vec{r}(\eta, \xi, \phi)}{dt} = U(\eta, \xi, \phi) = U_\eta \hat{\eta} + U_\xi \hat{\xi} + U_\phi \hat{\phi} \tag{18a}$$

The potential energy is given by

$$V = M(\eta, \xi, \phi) \Phi(\eta, \xi, \phi) \tag{18b}$$

where M is given in (Musongong and Howusu, 2004). Then the Lagrangean for the system is given by

$$L = \left\{ \begin{array}{l} \frac{2\pi a^5 \rho_0 \xi_0}{3} \left(\xi_0^2 + 1\right) \left[\frac{\dot{\eta}^2\left(\xi^2 - \eta^2\right)}{\left(1 - \eta^2\right)} + \frac{\dot{\xi}^2\left(\xi^2 - \eta^2\right)}{\left(\xi^2 - 1\right)} + \dot{\phi}^2\left(1 - \eta^2\right)\left(\xi^2 - 1\right) \right] \\ -\left(\frac{4\pi\rho_0 a^3}{3\sqrt{a}} \right)^2 G \left[\frac{\xi_0 Q_0(\xi) P_0(\eta)}{\left\langle \frac{d}{d\xi} Q_0(\xi) \right\rangle_{\xi=\xi_0}} + \frac{Q_2(\xi) P_2(\eta) \left\langle \frac{d}{d\xi} P_2(\xi) \right\rangle_{\xi=\xi_0}}{3\left[Q_2(\xi) \left\langle \frac{d}{d\xi} P_2(\xi) \right\rangle_{\xi=\xi_0} - P_2(\xi) \left\langle \frac{d}{d\xi} Q_2(\xi) \right\rangle_{\xi=\xi_0} \right]} \right] \end{array} \right\} \tag{19}$$

Consider the prolate field as a conservative system whose potential energy does not depend on the generalised velocities so that the Lagrange's equation for the system is also conserved. Hence the equations of motion are then given by:-

$$0=\left\{\begin{array}{l}\ddot{\eta}-\dot{\eta}\left[\left(\xi^2-\eta^2\right)\left(\xi^2-1\right)\left(1-\eta^2\right)\right]^{-1}\left[\dot{\eta}^2\left(\xi^2-1\right)\left(1-\xi^2\right)+\dot{\xi}^2\left(1-\eta^2\right)^2+\dot{\phi}^2\left(\xi^2-1\right)^2\left(1-\eta^2\right)^2\right]\\ -\dfrac{\left(1-\eta^2\right)K'B}{\left(\xi^2-\eta^2\right)}\left\{\dfrac{\xi_0 Q_0(\xi)\dot{P}_0(\eta)}{\left\langle \frac{d}{d\xi}Q_0(\xi)\right\rangle_{\xi=\xi_0}}+\dfrac{Q_2(\xi)\dot{P}_2(\eta)\left\langle \frac{d}{d\xi}P_2(\xi)\right\rangle_{\xi=\xi_0}}{3\left[Q_2(\xi)\left\langle \frac{d}{d\xi}P_2(\xi)\right\rangle_{\xi=\xi_0}-P_2(\xi)\left\langle \frac{d}{d\xi}Q_2(\xi)\right\rangle_{\xi=\xi_0}\right]}\right\}\end{array}\right\} \tag{20}$$

Also

$$0=\left\{\begin{array}{l}\ddot{\xi}-\dot{\xi}\left[\left(\xi^2-\eta^2\right)\left(\xi^2-1\right)\left(1-\eta^2\right)\right]^{-1}\left[\dot{\eta}^2\left(\xi^2-1\right)^2+\dot{\xi}^2\left(1-\eta^2\right)\left(\eta^2-1\right)+\dot{\phi}^2\left(\xi^2-1\right)^2\left(1-\eta^2\right)^2\right]\\ +\dfrac{\left(\xi^2-1\right)K'B}{\left(\xi^2-\eta^2\right)}\left\{\begin{array}{l}\dfrac{\xi_0\dot{Q}_0 P_0(\eta)}{\left\langle \frac{d}{d\xi}Q_0(\xi)\right\rangle_{\xi=\xi_0}}+\dfrac{\dot{Q}_2(\xi)P_2(\eta)\left\langle \frac{d}{d\xi}P_2(\xi)\right\rangle_{\xi=\xi_0}}{3\left[Q_2(\xi)\left\langle \frac{d}{d\xi}P_2(\xi)\right\rangle_{\xi=\xi_0}-P_2(\xi)\left\langle \frac{d}{d\xi}Q_2(\xi)\right\rangle_{\xi=\xi_0}\right]}\\ -\dfrac{Q_2(\xi)P_2(\eta)\left\langle \frac{d}{d\xi}P_2(\xi)\right\rangle_{\xi=\xi_0}\left[\dot{Q}_2(\xi)\left\langle \frac{d}{d\xi}P_2(\xi)\right\rangle_{\xi=\xi_0}-\dot{P}_2(\xi)\left\langle \frac{d}{d\xi}Q_2(\xi)\right\rangle_{\xi=\xi_0}\right]}{9\left[Q_2(\xi)\left\langle \frac{d}{d\xi}P_2(\xi)\right\rangle_{\xi=\xi_0}-P_2(\xi)\left\langle \frac{d}{d\xi}Q_2\right\rangle_{\xi=\xi_0}\right]}\end{array}\right\}\end{array}\right\} \tag{21}$$

and

$$0=\ddot{\phi}\left(1-\eta^2\right)\left(\xi^2-1\right)B^{-1} \tag{22}$$

where

$$B^{-1}=\frac{4\pi a^5\rho_0\left(\xi_0^2+1\right)}{3} \tag{23}$$

and

$$K'=\left(\frac{4\pi\rho_0 a^3}{3\sqrt{a}}\right)^2 G \tag{24}$$

Azimuthal solution: From (22) we obtained the azimuthal solution as

$$\dot{\phi}=\frac{l\,B}{\left(1-\eta^2\right)\left(\xi^2-1\right)} \tag{25}$$

where l is constant of the motion. Equation 25 is the equation for conservation of angular momentum for particles in the gravitational field of a massive prolate spheroidal body. This could be compared to that of spherical body given by:-

$$\dot{\phi}=\frac{l}{r^2\sin^2\theta} \tag{26}$$

which is well known.

Speed of a particle in the above field: From (20) for a particle moving along the η-component at the origin we have that

$$\xi=\xi_0,\qquad \dot{\xi}=\dot{\phi}=0 \tag{27}$$

$$\ddot{\eta}-\dot{\eta}^3\frac{\left(1-\xi_0^2\right)}{\left(1-\eta^2\right)}=\left(1-\eta^2\right)K'B\left[L^2\dot{P}_0(\eta)+T^2\dot{P}_2(\eta)\right] \tag{28}$$

where

$$L^2=\frac{\xi_0 Q_0(\xi)}{\left\langle \frac{d}{d\xi}Q_0(\xi)\right\rangle_{\xi=\xi_0}} \tag{29}$$

and

$$T^2 = \frac{Q_2(\xi)\left\langle \frac{d}{d\xi} P_2(\xi) \right\rangle_{\xi=\xi_0}}{3\left[Q_2(\xi)\left\langle \frac{d}{d\xi} P_2 \right\rangle_{\xi=\xi_0} - P_2(\xi)\left\langle \frac{d}{d\xi} Q_2(\xi) \right\rangle_{\xi=\xi_0} \right]} \quad (30)$$

Both L^2 and T^2 are uniquely defined hence transforming (28) we have:-

$$\frac{dw(\eta)}{d\eta} - w^2 \frac{\left(1-\xi_0^2\right)}{\left(1-\eta^2\right)} = \left(1-\eta^2\right) \quad (31)$$
$$K'B\left[L^2\dot{P}_0(\eta) + T^2\dot{P}_2(\eta)\right]w^{-1}$$

or

$$\frac{df(\eta)}{d\eta} + R(\eta)f = S(\eta) \quad (32)$$

where

$$R(\eta) = -\frac{\left(1-\xi_0^2\right)}{\left(1-\eta^2\right)} \quad (33)$$

and

$$S(\eta) = \left(1-\eta^2\right)K'B\left[L^2\dot{P}_0(\eta) + T^2\dot{P}_2(\eta)\right] \quad (34)$$

Equation (32) is linear and hence its solution is given in (Riley, 1974) as

$$f(\eta) = K'B\left[\left(\frac{1-\eta}{1+\eta}\right)^{-\frac{1}{2}} \exp\left(1-\xi_0^2\right)\right] \quad (35)$$
$$\left\{\int\left(1-\eta^2\right)\left[L^2\dot{P}_0(\eta) + T^2\dot{P}_2(\eta)\right] \left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}} \exp^{-\left(1-\xi_0^2\right)} d\eta + C\right\}$$

or

$$f(\eta) = \left(\frac{1-\eta}{1+\eta}\right)^{-\frac{1}{2}}$$
$$\left\{\int \frac{\left(1-\eta^2\right)(1-\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}} \left[L^2\dot{P}_0(\eta) + T^2\dot{P}_2(\eta)\right] d\eta\right\} + A \quad (36)$$

where A in (36) is a new constant and depending on the boundary conditions available at a particular time. Given a particular situation with (η, ξ, ϕ) as explicit define parameters the evaluation of (36) has very many yet undiscovered corrections terms and a refined solution for a prolate spheroidal massive body.

Practical application to rain-drop: For a prolate spheroidal point like body, the potential outside vanishes at the boundary so that a nontrivial solution of (16) exists and the first term in (16) has no contribution as the derivative is taken at $\xi = \xi_0$. We evaluated the harmonics involved in (16) while allowing the radius a to vary from end to end of the body, then

$$\frac{1}{a^2} \approx x_{n+1} \quad (37)$$

For simplifications we allowed motion only in the vertical direction (harmonics) while keeping the motion in the horizontal direction constant in time. That is

$$\left.\begin{array}{l} \frac{d}{d\xi} P_{2(\xi)} = x_n \\ , \\ \frac{d}{d\xi} Q_{2(\xi)} = b \end{array}\right\} \quad (38)$$

where b is constant and n is an integer.
Using (37) and (38) we transform (16) to a nonlinear differential equation as

$$x_{n+1} = \frac{Kx_n}{b + x_n} \quad (39)$$

where b, and K > 0. We let

$$\bar{x} = x_{n+1} = x_n \quad (40)$$

so that (39) becomes

$$\bar{x}\left(\bar{x} + b - K\right) = 0 \quad (41)$$

or

$$\bar{x} = K - b, \quad \bar{x} = 0 \quad (42)$$

The steady state will make sense only if K > b, since a negative value will mean that the massive body is contracting. For a small deviation we have

$$x'_{n+1} = rx'_n \quad (43)$$

where r is some function and

$$\begin{aligned} r &= \left.\frac{df}{dx}\right|_{x} \\ &= \frac{d}{dx}\left(\frac{Kx}{b+x}\right)\Bigg|_{\bar{x}} \\ &= \frac{Kb}{\left(b+\bar{x}\right)^2} \\ &= \frac{b}{K} \end{aligned} \quad (43)$$

Hence Eq. 43 shows that the bifurcation values are points that demarcate the abrupt change in the structure of the raindrop as it transverse $(-\infty, \infty)$ in its motion for instance. The rain-drop therefore becomes stable under (43) when $\frac{b}{K}$ is not greater than unity acknowledging the fact that the body is not expanding after condensation (grow in size as it moves through the atmosphere). When $\frac{b}{K}$ is greater than unity implies that the drop has absorbed energy from the environment. This is hardly the case. When $\frac{b}{K}$ is less than unity the size of the raindrop shrinking and the resulting collision with atmospheric particles increases thermal activities of the atmosphere bringing about global warming. This is another source of global warming of the atmosphere even though there are rain particles in the atmosphere. On this note we attribute warming of the atmosphere as a geometrical consideration.

CONCLUSION

It is worth noting that the likes of Vinti, Hele and Merson, O'keefe, Eckels and Squires, Garfinkel etc. have been investigating the motion of an earth satellite because they are faced with a natural problem that required a solution just the same the one of global warming. Equation (36) is an implicit expression for the speed of a particle in the field of a prolate spheroid moving with a speed compared to that of well known spherical body. Equations (25) and (36) are very new and have no analogue hitherto been found in nature. This could be applied to atmospheric particles, plasma (gas) particles etc. other than rain-drops for a plausible result. It is expected that due to the geometry of the Earth global warming is a geometrical problem than environmental pollution. Also with this result an appeal is made to experimentalists (plasma and atmospheric physics inclusive) as a matter of urgency to investigate all motion of planets, rain-drops, comets etc., as has been carried out theoretically in this study.

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