Journal of Engineering and Applied Sciences 2 (2): 350-353, 2007

Effect of Surface Tension on Two-Dimensional Free Surface Flow

F. Guechi and H. Mekias
Department of Mathematics, Faculty of Sciences, Ferhat Abbas University, Setif 19000, Algeria

Abstract: Water or fluid flowing on a horizontal half-plane can either detach from the surface when reaching the boundary or it can continue flowing on the opposite side of the half-plane. The latter phenomenon is called the "teapot effect"; it has been the subject of many investigations over the past decades. Here a two dimensional flow exhibiting a teapot effect is considered without neglecting the surface tension. The problem is solved numerically via a series truncation method for various values of the Weber number, á. We find solutions for α greater than or equal to 10 and no solutions for α less than 10.

Key words: Phenomenon, teapot effect, truncation

INTRODUCTION

When a liquid is poured from a container, it sometimes runs down along the under side of the spout rather than falling as a free stream. This phenomenon called the "teapot effect" was studied by many authors (Reiner, 1956; Keller, 1957; Kistler and Scriven, 1994). Reiner (1956) studied the phenomenon experimentally and concluded that it is neither due to surface tension nor to adhesion of the liquid to the container surface. Keller (1957) showed that it is explained by the Bernoulli principle that the pressure is low where the velocity is high, so that the atmospheric pressure pushes the flowing fluid against the bottom of the spout. Keller gave an exact solution of the problem when neglecting gravity and surface tension. Scheidegger (1970) took the idea of the teapot effect and explained the formation of hoodoos and made interesting calculation considering the flowing of water. Daboussi *et al.* (1998), Baines and Whithead (2003) calculated the flow numerically over obstacles of various configurations.

In this note, we approximate this natural phenomenon by considering a two-dimensional flow over a semi infinite horizontal plate. We neglect gravity and take into account the effect of surface tension. The flow is taken to be irrotational, incompressible and inviscid. The flow is bounded by the free surface ABC and the horizontal wall O'OO' Fig. 1. Far downstream the flow is uniform with a constant velocity U and a constant depth D.

The flow is characterized by the Weber number:

$$\alpha = \frac{\rho U^2 D}{T} \qquad (1)$$

Where T is the surface tension and ρ is the density of the fluid.

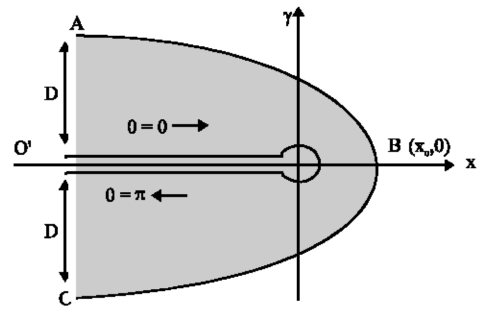

Fig. 1: The configuration of the flow. The x-axis is along the horizontal flat board and the y-axis is vertically through the tip system of the board

We compute accurate numerical solutions for the fully nonlinear problem via a truncation series technique (Mekias, 1991; Vandan, 1986a, 1986b; Bouderah and Mekias, 2002; Gasmi and Mekias, 2003) where the mesh points are only on the free surface. The problem is first formulated as an integral equation for the unknown shape of the free surfaces then it is transformed to solving an algebraic non linear system. This algebraic system is solved by Newton's method (Boumahrat and Gourdin, 1991). Our results show that there is a unique solution for each value $\alpha \geq 10$ and no solution for $\alpha < 10$.

Formulation: Let us consider a two-dimensional flow of inviscid irrotational and incompressible fluid over a semi infinite horizontal wall O'OO' and bounded by a free surface ABC Fig. 1.

We introduce cartesian coordinates with the origin at the edge of the plate, x-axis along the horizontal plate O'O and the y-axis perpendicularly to the x-axis through the edge of the plate. Far downstream, $x \to -\infty$, the flow approaches a uniform stream with a constant velocity U and a constant depth D.

Corresponding Author: F. Guechi, Department of Mathematics, Faculty of Sciences, Ferhat Abbas University, Setif 19000, Algeria

The problem is studied numerically neglecting the effect of gravity and considering the effect of surface tension. We define the dimensionless variables by choosing U as the unit velocity and D as the unit length.

We introduce the complex variable $\bar{z} = \bar{x} + i\bar{y}$, the complex potential f = ϕ+iψ and the complex velocity ξ = u+iv. Where ϕ is the potential function, ψ the streamline function and (u,v) are the components of the vector velocity $\vec{V}$ in the above system of coordinates. We know, from the potential flow theory that ξ = df/dz.

We define the function τ-iθ by the relation

$$\xi = \exp(\tau - i\theta) \quad (2)$$

Without loss of generality, we choose ϕ = 0 at $B(\bar{x}_0, 0)$ and ψ = 0 on the streamline O'OO'. The flow configuration in the complex potential plane (ϕ,ψ) is illustrated in Fig. 2.

On the free surface ABC, the Bernoulli equation yields

$$\bar{P} + \frac{1}{2}\rho \bar{q}^2 = \bar{P}_0 + \frac{1}{2}\rho U^2 \quad \text{on } \psi = 1 \text{ and } -\infty < \phi < +\infty \quad (3)$$

Where $\bar{P}$ and $\bar{q}$ are the fluid pressure and speed respectively, just inside the free surface. The right-hand side of Eq. 3 is evaluated from the condition far downstream.

A relationship between $\bar{P}$ and P_0 is given by Laplace's capillarity formula

$$\bar{P} - P_0 = TK \quad (4)$$

Here K is the curvature of the free surface and T the surface tension.

If we substitute (4) into (3), we obtain

$$\frac{1}{2}\bar{q}^2 - \frac{T}{\rho}K = \frac{1}{2}U^2 \quad (5)$$

We introduce the non-dimensional variables

$$q = \bar{q}/U,\ x = \bar{x}/D,\ y = \bar{y}/D,\ \alpha = \frac{\rho D U^2}{T}$$

In dimensionless variables (5) becomes

$$\exp(2\tau) + \frac{2}{\alpha}\exp(\tau)\left|\frac{\partial\theta}{\partial\phi}\right| = 1 \quad \text{on ABC} \quad (6)$$

Here α is the Weber number defined by (1).

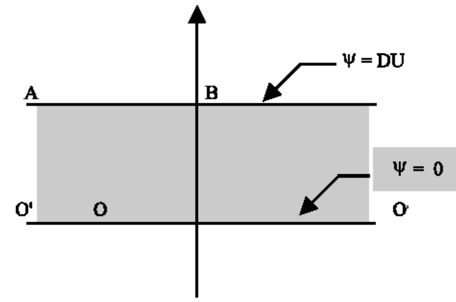

Fig. 2: The flow configuration in the complex potential plane

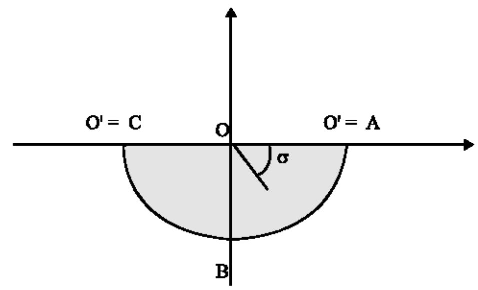

Fig. 3: The flow domain in the t-plane

The kinematics condition on O'O and OO' yields

$$\begin{cases} \theta = 0, & \psi = 0, \quad -\infty \prec \phi \prec 0 \\ \theta = -\pi, & \psi = 0, \quad 0 \prec \phi \prec \infty \\ \theta = -\frac{\pi}{2}, & \psi = 1, \quad \phi = 0 \end{cases} \quad (7)$$

This completes the formation of the problem. We seek τ-iθ as analytic function of f = ϕ+iψ in the strip 0<ψ<1 (Fig.2).

Numerical procedure: We define a new variable t by the relation

$$f = \frac{1}{\pi}\log\left[\left(\frac{t-1}{t+1}\right)^2\right] \quad (8)$$

This transformation maps the flow domain into the lower half of the unit circle in the complex t plane Fig. 3.

From the potential flow theory, the function ξ and f have no singularities in the flow domain except at O where we have a flow around an angle of 360°. Local asymptotic analysis gives

$$\xi = O\left(\frac{1}{t}\right) \quad \text{as} \quad t \to 0$$

Hence, we define a function $\Omega(t)$ by the relation

$$\exp(\tau - i\theta) = \frac{1}{t}\Omega(t) \tag{9}$$

The function $\Omega(t)$ is bounded and continuous on the unit circle and analytic in the interior-disk, thus $\Omega(t)$ can be expanded in the form of a Taylor expansion in even powers of t. Hence,

$$\exp(\tau - i\theta) = \frac{1}{t}\exp(\sum_{n=0}^{\infty} a_n\, t^{2(n-1)}) \tag{10}$$

The function (10) satisfies (7) if all the coefficients a_n are real. The coefficients a_n have to be determined to satisfy (6).

We use the notation $t = |t|\exp(i\sigma)$ so that points on ABC are given by $t = \exp(i\sigma)$ Using (8), (6) can be written in the form

$$\exp(2\bar{\tau}) \quad + \frac{\pi}{\alpha}\exp(\bar{\tau})\left|\sin(\sigma)\frac{\partial\bar{\theta}}{\partial\sigma}\right| = 1 \tag{11}$$

Here $\bar{\tau}(\sigma)$ and $\bar{\theta}(\sigma)$ denote the values of τ and θ on the free surface ABC.

We solve the problem by truncating the infinite series in (10) after N terms. We find the N coefficients a_n by collocation. Thus we introduce the N mesh points

$$\sigma_I = \frac{-\pi}{2N}\left(I - \frac{1}{2}\right) \qquad I = 1,\ldots,N \tag{12}$$

Using (10) we obtain $\left[\bar{\tau}(\sigma)\right]_{\sigma=\sigma_I}$, $\left[\bar{\theta}(\sigma)\right]_{\sigma=\sigma_I}$ and $\left[\frac{\partial\bar{\theta}}{\partial\sigma}\right]_{\sigma=\sigma_I}$ in terms of coefficients a_n. Substituting these expressions into (11) at the point σ_I we obtain N nonlinear algebraic equations for the N unknowns a_n, n =1,...,N We solve this system by Newton's method for a given value of α (here, α is a parameter). The shape of the free surface is obtained by integrating numerically the relations:

$$\begin{cases} \dfrac{\partial\tilde{x}}{\partial\sigma} = \exp(-\tau)\cos(\theta)\dfrac{\partial\phi}{\partial\sigma} \\ \dfrac{\partial\tilde{y}}{\partial\sigma} = \exp(-\tau)\sin(\theta)\dfrac{\partial\phi}{\partial\sigma} \end{cases} \tag{13}$$

Here $\tilde{x}$ and $\tilde{y}$ are the values of x and y on the free surface.

RESULTS AND DISCUSSION

In the absence of gravity and surface tension, the problem has an exact solution that we can compute using the hodograph transformation of the free stream line theory due to Birkhof (1967). The explicit form of the free surface in x-y plane is given by

$$\begin{cases} x = x_0 + \dfrac{D}{\pi}\log(1-(\cos\theta)^2) \\ y = \dfrac{2D}{\pi}(\dfrac{\pi}{2}+\theta) \end{cases} \tag{14}$$

The turning point of the free surface (y = 0) is at the ordinate x = b = 0.429 Fig. 4.

In presence of surface tension and (or) force of gravity, there is no exact known solution. In our study we only consider surface tension to evaluate its effect on the flow. Applying the numerical procedure described in section 3, we compute the solution for various values of the surface tension. We note that the surface tension is evaluated through the dimensionless parameter α (the Weber number). For a fixed value of Weber number α the coefficients a_n of the series $\sum_{n=0}^{\infty} a_n\, t^{2(n-1)}$ were found to decrease very rapidly Table 1.

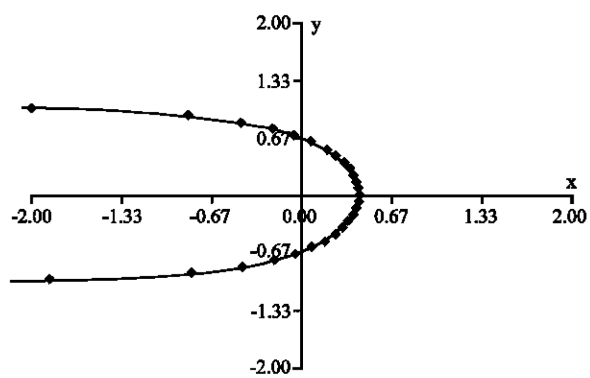

Fig. 4: Free surface flow for $\alpha \geq 150$ (■ α = 150, ★α = 500, ____α = 800)

Table1: The series coefficients for different values of α

α	a_1	a_{10}	a_{20}	a_{30}	a_{40}
100	$-1.005\ 10^{-2}$	$5.930\ 10^{-5}$	$1.199\ 10^{-5}$	$3.831\ 10^{-6}$	$3.174\ 10^{-7}$
60	$-1.680\ 10^{-2}$	$9.738\ 10^{-5}$	$2.012\ 10^{-5}$	$6.410\ 10^{-6}$	$7.426\ 10^{-7}$
20	$-5.129\ 10^{-2}$	$2.811\ 10^{-4}$	$5.423\ 10^{-5}$	$1.303\ 10^{-5}$	$2.861\ 10^{-6}$
10	$-1.053\ 10^{-1}$	$4.620\ 10^{-4}$	$1.121\ 10^{-4}$	$1.118\ 10^{-5}$	$4.450\ 10^{-7}$

Table 2: Comparison between the coefficients of the series $\sum_{n=0}^{\infty} a_n\, t^{2(n-1)}$ and $\sum_{n=0} \frac{1}{(1+0.1)^n}$ for $\alpha = 10$

i	1	10	20	30	40
$\alpha(i)$	$1.05\ 10^{-1}$	$4.62\ 10^{-4}$	$1.12\ 10^{-4}$	$1.11\ 10^{-5}$	$4.45\ 10^{-7}$
$(1+0.1)^i$	$9.09\ 10^{-1}$	$3.85\ 10^{-1}$	$1.48\ 10^{-1}$	$0.57\ 10^{-1}$	$0.22\ 10^{-1}$

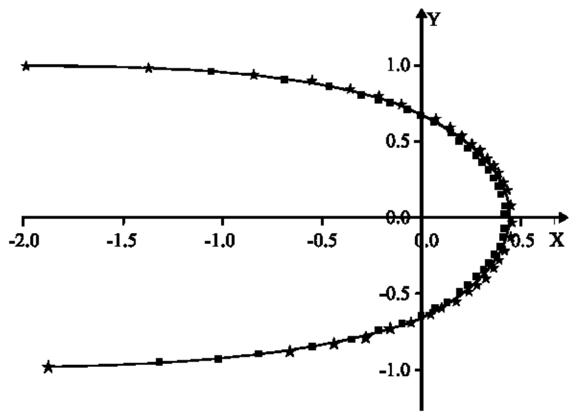

Fig. 5: Free surface shapes for different values of the Weber number (α≥10).

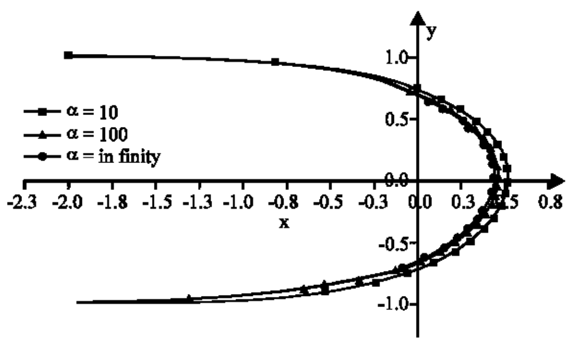

Fig. 6: Free surface configuration without surface tension, (____) Via analytical computation by free streamline theory and (♦) Via numerical integration using the present scheme

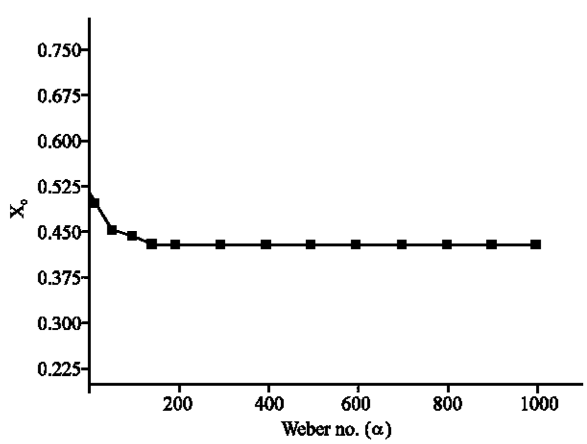

Fig. 7: The position of the turning point "B" versus Weber number α

Table 2 shows that the series $\sum_{n=0}^{\infty} a_n t^{2(n-1)}$ converges.

For α<10 the numerical scheme diverges and no solution is found. This is probably due to the fact that the surface tension tends to strengthen the surface as for the teapot effect tends to bend at the turning point. Hence, there must be a value of the surface tension (the Weber number) where the two effects are of the same importance. This can explain the fact that the scheme diverges.

For $\alpha \geq \alpha_0$ (α_0 = 150): All free surfaces for different values of $\alpha \geq \alpha_0$ are the same within graphical accuracy Fig. 5 and coincide with the graph of the exact solution without surface tension Fig. 4. This suggests that the surface tension can be neglected if $\alpha \geq \alpha_0$.

Fig. 6 shows the free surface shapes for different values of Weber number α≥10 where the turning point changes its position. The effect of the surface tension is more apparent on the position of the turning point B as shown in Fig. 7.

REFERENCES

Reiner, M., 1956. J. Phys. Today 9: 16.

Keller, J.B., 1957. J. Applied Phy., 28: 859.

Kistler, S.F. and L.E. Scriven, 1994. J. Fluid Mech., 263: 19-62.

Daboussi, D., F. Dias and J.M. Vanden Broeck, 1998. Eur. J. Mech. B/Fluids, 17: 19-31.

Baines, P.G. and J.A. Whitehead, 2003. Physics of Fluids, pp: 15.

Batchelor, G.K., 1967. An introduction to fluid dynamics, Cambridge.

Boumahrat, M. and A. Gourdin, 1991. Méthodes numériques appliquées, O.P.U.

Mekias, H., 1991. Two-dimensional free-surface due to a point singularity, Ph.D. thesis, University of Wisconsin-Madison.

Vanden-Broeck, J.M., 1986 a. Phys. Fluids, 29: 3148.

Vanden-Broeck, J.M., 1986 b. Phys. Fluids 29: 2798.

Bouderah, B. and H. Mekias, 2002. Intern. J. Syst. Cybern., 31: 305-316.

Gasmi, A. and H. Mekias, 2003. J. Phys. A: Mat. Gen., 36: 851-862.

Scheidegger, A.E., 1970. Theoritical Geomorphology. Springer-Verlag, Berlin, Heidelberg, New York, 2nd Edn., pp: 405-408.