Journal of Engineering and Applied Sciences 2 (2): 319-330, 2007

Probabilistic Transformation Method in Reliability Analysis

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Abstract: In this study, a proposed technique is presented in order to evaluate the stochastic mechanical response. This technique is based on the combination of the probabilistic transformation methods and the deterministic Finite Element Method (FEM). The transformation technique evaluates the Probability Density Function (PDF) of the system output by multiplying the input PDF by the Jacobean of the inverse mechanical function. This approach has the advantage of giving directly the whole density function of the response -in closed form-, which is very helpful for reliability analysis.

Key words: Finite element method, method of the unit load, FORM, probabilistic methods, reliability analysis, sampling, sensitivity, simulation, transformation method

INTRODUCTION

Mechanical modeling of physical systems is often complicated by the presence of uncertainties. The implications of these uncertainties are particularly important in the assessment of several potential regulatory options. Even though significant effort may be needed to incorporate uncertainties into the modeling process, this could potentially result in providing useful information that can aid in decision making.

For several decades, the theory of probability has been used in mechanics to model the random structural properties (materials, geometry, boundary conditions...) and phenomena (turbulence, seismic wave, loads...) acting on the mechanical systems. The probabilistic approach takes into account the uncertainties on the model data in order to improve the robustness of the forecasts and optimized configuration. The structural reliability has become a discipline of international interest, addressing issues such as the performance-based cost-safety balancing (Procacccia and Morilhat, 1996).

In this research, a proposed technique is presented in order to evaluate the stochastic mechanical response. The method is based on the combination of the Probabilistic Transformation Methods (PTM) for a random variable (e.g., Young's modulus or load) and the deterministic Finite Element Method (FEM). The transformation technique evaluates the Probability Density Function (PDF) of the system output by multiplying the input PDF by the Jacobean of the inverse mechanical function.

SENSITIVITY/UNCERTAINTY ANALYSIS METHODS

Conventional methods for sensitivity analysis and uncertainty propagation can be broadly classified into four categories: "sensitivity checking", analytical methods, sampling based methods, and computer algebra based methods.

Sensitivity checking involves the study of the model response for a set of changes in model formulation, and for selected parameter combinations. Analytical methods involve either the differentiation of model equations and subsequent solution of a set of auxiliary sensitivity equations, or the reformulation of original model using stochastic differential equations. On the other hand, the sampling based methods involve running the original model for a set of input parameter combinations and estimating the sensitivity/uncertainty using the model outputs at those points. Another sensitivity method is based on direct manipulation of the computer program, and is termed as automatic differentiation.

Sensitivity checking methods: In this study (Sistla *et al.*, 1991), the model is run for a set of sample points of the concerned parameters with straightforward changes in structural model. This approach is often used to evaluate the robustness of the model, by testing whether the model response changes significantly in relation to changes in model parameters.

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The primary advantage of this approach is that it accommodates both qualitative and quantitative information regarding variation in the model. However, the main disadvantage is that detailed information about the uncertainties are difficult to obtain using this approach. Furthermore, the sensitivity information obtained depends to a great extent on the choice of the sample points, especially when a small number of simulations can be performed.

Analytical methods: Some of the widely used analytical methods for sensitivity/uncertainty are: differential analysis methods, Green's function method, spectral stochastic finite element method, and coupled and decoupled direct methods.

The analytical methods require access to the governing model equations and may involve writing additional computer code for the solution of the auxiliary equations, which may be impractical and sometimes impossible. For example, reformulating an existing computational model developed by others could require prohibitive amounts of resources.

Differential analysis methods: Differential analysis methods include the Neumann expansion (Tatang, 1992) and the perturbation method (Tatang, 1995). The Neumann expansion method involves finding the inverse of the model operator through the expansion of the model equations, and hence it has limitations on the type of model equations it can address. The perturbation method involves expansion of model output as a series of small random perturbations in model parameters, followed by the solution of the series coefficients. The Neumann expansion and the perturbation based methods have been applied in the design and uncertainty analysis of mechanical structures. The main limitation of these methods is that the perturbation should be small. Further, these methods are in general difficult to apply to complex and nonlinear systems, as the model equations are often mathematically intractable.

Green's function method: In the Green's function method (Doughtery and Rabitz, 1979), the sensitivity equations of a model are obtained by differentiating the model equations. The sensitivity equations are then solved by constructing an auxiliary set of Green's functions. This method minimizes the number of differential equations to be solved for sensitivity. They are then replaced by integrals that can be easily evaluated.

Spectral stochastic finite element method: This method relies on the use of stochastic processes in terms of a series expansion, specifically the Karhunen-Loeve expansion (Ghanem and Spanos, 1991). For finite element problems, this approach results in a set of linear matrix equations in terms of a random vectors. The matrix equations are solved either using operator expansions or by using the Galerkin's method. One of the main features of this method is the representation of random parameters in terms of orthogonal functions of standardized random variables; the expansion is also known as "polynomial chaos expansion" and forms the basis for the development of the "Stochastic Response Surface Method" (SRSM).

Sampling based methods: Sampling based methods do not require the access to model equations or even to the model code. These methods involve running a set of sample points, and establishing a relationship between input and output based on the model results at the sample points. Some of the widely used sampling based sensitivity/uncertainty analysis methods are:

- Monte Carlo simulation.
- Fourier Amplitude Sensitivity Test (FAST)
- Reliability based methods
- Response surface methods.

Monte carlo simulation: Monte Carlo (MC) methods are the most widely used means for uncertainty analysis, with many applications. These methods involve random sampling from the distribution of input and successive model runs until a statistically significant distribution of output is obtained. They can be used to solve problems with physical probabilistic structures, such as uncertainty propagation in models or solution of stochastic equations, or can be used to solve non-probabilistic problems, such as finding the area under a curve. Monte Carlo methods are also used in the solution of problems that can be modeled by the sequence of a set of random steps that eventually converge to a desired solution. Problems such as optimization and the simulation of movement of fluid molecules are often addressed through Monte Carlo simulations (Rubinstein, 1981; Sobol, 1994).

Since these methods require a large number of samples (or model runs), their applicability is sometimes limited to simple models. In study of computationally intensive models, the time and resources required for these methods could be prohibitively expensive. A degree of computational efficiency is accomplished by the use of Modified Monte Carlo (MMC) methods that sample from the input distribution in an efficient manner, so that the number of necessary runs compared to the simple Monte Carlo method is significantly reduced.

Fourier Amplitude Sensitivity Test (FAST): Fourier Amplitude Sensitivity Test (FAST) is a method based on Fourier transformation of uncertain model parameters into a frequency domain, thus reducing the multi-dimensional model into a single dimensional one. For a model with m model parameters, k_1, k_2,....k_m and n outputs, u_1, u_2,....u_m, such that $u_i = f_i(t;k_1, k_2,....k_m)$ I = 1,2,...n the FAST (McRae *et al*., 1982) method involves the transformation (Koda *et al*., 1979) of the parameters into a frequency domain spanned by a scalar s, as follows:

$$k_l = G_l(\sin w_l s), \quad l = 1,2,...,m$$

The outputs are then approximated as:

$$\bar{u}_i(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_i(t;k_1(s),k_2(s),...,k_m(s))ds$$

$$\sigma_i^2(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_i^2(t;k_1(s),k_2(s),...,k_m(s))ds - \bar{u}_i^2$$

These integrals are evaluated by repeatedly sampling the parameter space of s, which corresponds to the sampling in the multidimensional model parameter space.

Response surface methods: The response surface methods consist of following

- Screening to determine a subset of important model input parameters.
- Making multiple runs of the computer model using specific values and pairings of these input parameters
- Fitting a general polynomial model to the model data (using the method of least squares).

This fitted response-surface is then used as a replacement for the computer model, and all inferences related to sensitivity/uncertainty analysis for the original model are derived from this fitted model. This approach is sometimes termed as a "secondary model technique". El-Tawil *et al.* (1990, 1991) describe the adaptive nature of these methods.

RELIABILITY ANALYSIS

First and Second Order Reliability Methods (FORM and SORM, respectively) are approximation methods that estimate the probability of an event under consideration (typically termed "failure"). In study, these methods provide the contribution to the probability of failure from each input random variable, at no additional compu-tational effort. These methods are useful in uncertainty analysis of models with single failure criterion.

For a model with random parameters

$$x = (x_1, x_2,...,x_n)$$

and a failure condition

$$g(x_1, x_2,...,x_n) < 0$$

the objective of the reliability based approach is to estimate the probability of failure. In case of limit state of displacement, the failure condition can be defined as

$$g(x) = D_l - D(x) < 0$$

where D_l is a limit displacement at a location of interest.

If the joint probability density function for the set x is given by f_x, then the probability of failure is given by the n-fold integral:

$$P_f = P\{g(x) < 0\} = P\{D_l < D(x)\} = \int_{g(x)<0} f_x dx$$

where the integration is carried out over the failure domain. The evaluation of this integral becomes computationally demanding as the number of random variables (the dimension of the integration) increases; in fact if m is the number of function calls of the integrand per dimension and n is the dimension, the computation time grows as m^n (Hohenbichler *et al*., 1987). In addition, since the value of the integrand is small, the numerical inaccuracies can be considerably magnified when integrated over a multi-dimensional space (Baldocchi *et al*., 1995).

FORM and SORM use analytical schemes to approximate the probability integral, through a series of the following simple steps, as illustrated by Bjerager (1990):

- Mapping the basic random variables x and the failure function g(x), into a vector of standardized and uncorrelated normal variates u, as x(u) and G(u) respectively,
- Approximating the function G(u) by a tangent (FORM) or a paraboloid (SORM) at a failure point u^* closest to the origin, and
- Calculating the probability of failure as a simple function of u^*.

These methods are reported to be computationally very efficient compared to Monte Carlo methods, especially for scenarios corresponding to low probabilities of failure. Further, SORM is more accurate than FORM, but computationally more intensive, since it involves a higher order approximation.

The main drawbacks of FORM and SORM are that the mapping of the failure function on to a standardized set, and the subsequent minimization of the function, involve significant computational effort for nonlinear black box numerical models. In addition, simultaneous evaluation of probabilities corresponding to multiple failure criteria would involve significant additional effort. Furthermore, these methods impose some conditions on the joint distributions of the random parameters, thus limiting their applicability.

PROPOSED METHODS (PTM)

The theory of the Probabilistic Transformation Method or PTM is based on the following theorem (Soong, 1993).

Theorem: Suppose that X is a random variable with PDF (probability density function) f(x) and $A \subset \Re$ is the one–dimensional space where f(x)>0. Consider the random variable (function of x) Y = u(X), where y = u(x) defines a one-to-one transformation that maps the set A onto a set $B \subset \Re$ so that the equation y = u(x) can be uniquely solved for x in terms of y, say x = w(y). Then, the PDF of Y is:

$$g(y) = f\left[w(y)\right]|J|, \qquad y \in B,$$

where, $J = \frac{dx}{dy} = \frac{dw}{dy}$ is the Jacobian of the transformation.

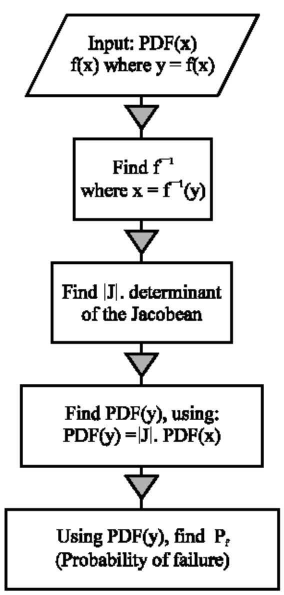


Fig. 1: Algorithm of the proposed method

The proposed technique is a combination of the deterministic finite element method and the random variable transformation technique. In this technique, the differential equation is solved firstly using the deterministic theory of finite element which yields to accurate nodal exact solutions. These solutions are then used to obtain the approximate PDF using the random variable transformation between the input random variables and the output variable. The accuracy of the solution is increased when increasing the number of elements as usual. The algorithm of this method is shown in Fig. 1.

APPLICATIONS

In the first application, we are going to analyze the reliability of a cantilever beam (Fig. 2) with random parameters (Young modulus E and the distributed load W).

FEM modeling the beam with 2 elements: The stiffness matrix of an element is given by Chateauneuf (2005).

For an element of n nodes, the deformation and the bending stress are given by:

$$\varepsilon(x) = -u\frac{d^2v(x)}{dx^2}$$

$$\sigma(x) = E.\varepsilon(x)$$

Let l = L/2, $u_1 = 0$ and $\theta_1 = 0$, the assembly of two elements leads to the following system:

$$\frac{8EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & -12L & -12 & 6L \\ 6L & 2L & -12L & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & 6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} T + \frac{wL}{3} \\ M + \frac{wL^2}{48} \\ \frac{wL}{3} \\ \frac{wL^2}{48} \\ \frac{wL}{3} \\ \frac{wL^2}{48} \end{bmatrix}$$

After simplification, the displacement of third node:

$$u_3 = \frac{WL^4}{8EI}$$

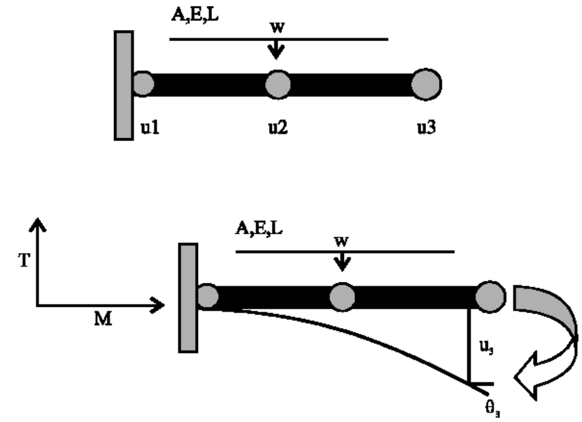


Fig. 2: Cantilever beam

Probabilistic study of u_3

Case 1: E uniformly distributed ➔U(10^8, 3.10^8)

Using the proposed technique ➔

$$\mathrm{PDF}(u_3) = |J|\,\mathrm{PDF}(E) = \frac{wL^4}{8Iu_3^2}\mathrm{PDF}(E)$$

$$= \begin{cases} \dfrac{WL^4}{16.10^8\, Iu_3^2} & \text{if } \dfrac{wL^4}{24.10^8\, I} \le u_3 \le \dfrac{wL^4}{8.10^8\, I} \\ 0 & \text{if not} \end{cases}$$

Reliability analysis: Let us suppose the limit displacement is u31 = L/180 = 0.0556, it is requested to find the failure probability $P_f = P(u \ge u_{31})$.

Numerical values:

$w = 12\,\mathrm{kN/m}$

$L = 10\,\mathrm{m}$

$I = 1{,}8.10^{-3}$

➔

$$P_f = \int_{0.0556}^{\infty} \mathrm{PDF}(u_3) = \int_{0.0556}^{\infty} \frac{WL^4}{16.10^8\, Iu_3^2}\, du_3$$

$$= \int_{0.0556}^{0.0833} \frac{12.10^4}{16.10^8.1{,}8.10^{-3}.u_3^2}\, du_3 = \frac{1}{4} = 0.25$$

Comparison with monte carlo

	Proposed method	Monte carlo simulation(10000)
P_f	0.25	0.2458

Case 2: W normally distributed ➔N (12,1)

Using the proposed technique

➔

$$\mathrm{PDF}(u_3) = \left\{ \frac{8EI}{L^4\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{8EIu_3}{L^4} - 12\right)^2} \right.$$

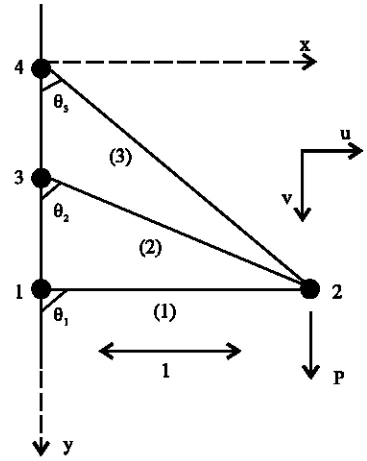


Fig. 3: Three-bar truss structure

Reliability analysis: Let us suppose now the limit displacement is u_{31} = L/220 = 0.0455, it is requested to find the failure probability $P_f = P(u \ge u_{31})$.

Numerical values:

$E = 2.10^8\,\mathrm{kN/m}$

$L = 10\,\mathrm{m}$

$I = 1{,}8.10^{-3}$

➔

$$P_f = \int_{0.0455}^{\infty} \mathrm{PDF}(u_3) = \int_{0.0455}^{\infty} \frac{8EI}{L^4\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{8EIu_3}{L^4} - 12\right)^2} du_3$$

$$= \int_{0.0455}^{0.0546} \frac{16.1{,}8.10^5}{10^4\sqrt{2.3{,}14}}\, e^{-\frac{1}{2}\left(\frac{16.1{,}8.10^5 u_3}{10^4} - 12\right)^2} du_3 = \frac{1}{4} = 0.1347$$

Comparison with monte carlo

	Proposed method	Monte carlo simulation(10000)
P_f	0.1347	0.1328

In the second application, we are going to analyze the reliability of a three-bar truss structure (Fig. 3) with random parameters (Young modulus E or the distributed load P).

FEM modeling the three-bar truss: The Stiffness matrix, in global axis, is given:

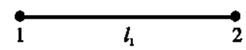
$$\left[K^e\right]^{(i)} = \frac{A^i E^i}{l_i} \begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix}$$

Where:

$$\begin{cases} i = number\ of\ element \\ A = cross\ section \\ E = Young's\ modulus \\ l = length\ of\ bar \\ \lambda = \cos\alpha \\ \mu = \sin\alpha \end{cases}$$

$\alpha = angle\ between\ the\ element\ and\ the\ horizontal$

Element (1): 1-2



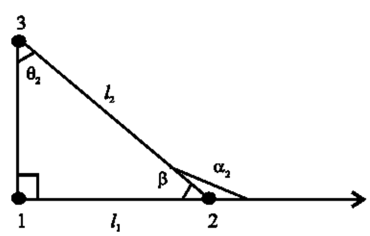
$$\begin{cases} \lambda = \cos\alpha_1 = 1 \\ \mu = \sin\alpha_1 = 0 \end{cases}$$

The stiffness matrix:

$$\left[K^e\right]^{(1)} = \frac{AE}{l_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To simplify, we suppose $A^i = A$, $E^i = E$.

Element (2): 2-3



$$\alpha_2 = \pi - \beta = \frac{\pi}{2} - \theta_2$$

$$\lambda = \cos\alpha_2 = \cos(\frac{\pi}{2} - \theta_2) = \sin\theta_2 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

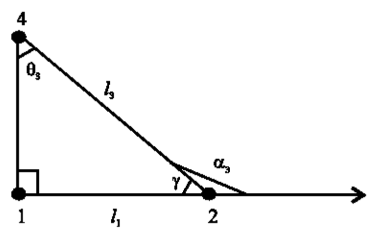
$$\mu = \sin\alpha_2 = \sin(\frac{\pi}{2} - \theta_2) = \cos\theta_2 = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$l_2 = \frac{l_1}{\cos\beta} = \frac{2l_1}{\sqrt{3}}$$

$$\rightarrow \left[K^e\right]^{(2)} = \frac{AE}{l_2}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

$$\rightarrow \left[K^e\right]^{(2)} = \frac{AE}{l_1}\begin{bmatrix} \frac{3\sqrt{3}}{8} & \frac{3}{8} & -\frac{3\sqrt{3}}{8} & -\frac{3\sqrt{3}}{8} \\ \frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{3}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{3\sqrt{3}}{8} & -\frac{3}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{\sqrt{3}}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} \end{bmatrix}$$

Element (3): 2-4



$$\alpha_3 = \pi - \gamma = \frac{\pi}{2} - \theta_3$$

$$\lambda = \cos\alpha_3 = \cos(\frac{\pi}{2} - \theta_3) = \sin\theta_3 = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\mu = \sin\alpha_3 = \sin(\frac{\pi}{2} - \theta_3) = \cos\theta_3 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$l_3 = \frac{l_1}{\cos\gamma} = \frac{2l_1}{\sqrt{2}}$$

$$\rightarrow \left[K^e\right]^{(3)} = \frac{AE}{l_3}\begin{bmatrix} \frac{2}{4} & \frac{2}{4} & -\frac{2}{4} & -\frac{2}{4} \\ \frac{2}{4} & \frac{2}{4} & -\frac{2}{4} & -\frac{2}{4} \\ -\frac{2}{4} & -\frac{2}{4} & \frac{2}{4} & \frac{2}{4} \\ -\frac{2}{4} & -\frac{2}{4} & \frac{2}{4} & \frac{2}{4} \end{bmatrix}$$

$$\rightarrow \left[K^e\right]^{(3)} = \frac{AE}{l_1}\begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

Using the assembly theorem, the global stiffness matrix is:

$$[K]^g = \frac{AE}{l_1}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & \frac{8+3\sqrt{3}+2\sqrt{2}}{8} & \frac{3+2\sqrt{2}}{8} & \frac{-3\sqrt{3}}{8} & \frac{-3\sqrt{3}}{8} & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} \\ 0 & 0 & \frac{3+2\sqrt{2}}{8} & \frac{3+2\sqrt{2}}{8} & \frac{-3}{8} & \frac{-\sqrt{3}}{8} & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} \\ 0 & 0 & \frac{-3\sqrt{3}}{8} & \frac{-3}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & \frac{-3}{8} & \frac{-\sqrt{3}}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

Therefore the assembly of three elements leads to the following system

$$\{F\} = [K]^g.\{U\}$$

Where:

$$\{F\} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ 0 \\ P \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix}, \quad \{U\} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After resolution, the vertical displacement of node 2 is:

$$v_2 = 3.25\frac{Pl_1}{AE}$$

Probabilistic study of v_2
Case 1: E(daN/cm^2) uniformly distributed ➔U(5, 10)
Using our technique ➔

$$\mathrm{PDF}(v_2) = |J|\,\mathrm{PDF}(E) = \frac{3.25Pl_1}{Av_2^2}\mathrm{PDF}(E)$$
$$= \begin{cases} \frac{3.25Pl_1}{Av_2^2} & \text{if } \frac{3.25Pl_1}{10A} \le v_2 \le \frac{3.25Pl_1}{5A} \\ 0 & \text{if not} \end{cases}$$

Reliability analysis: Let us suppose now the limit displacement is v_{21} = 2, it is requested to find the failure probability Pf = P($v \ge v_{21}$).

Numerical values:

$P = 1.2$ dan

$l_1 = 3$ cm

$A = 1\ cm^2$

$$\rightarrow P_f = \int_2^{\infty} PDF(v_2) = \int_2^{\infty} \frac{3.25Pl_1}{Av_2^2} dv_2 = \int_2^{2.34} \frac{3.25\times 1.2\times 3}{5.v_2^2} dv_2 = \frac{17}{100} = 0.17$$

Comparison with monte carlo

	Proposed method	Monte carlo simulation(10000)
P_f	0.17	0.1681

Case 2: P normally distributed $\rightarrow N(1.2,0.9)$

Using our technique $\rightarrow$

$$PDF(v_2) = |J| PDF(E) = \frac{AEv_2}{3.25l_1} PDF(P)$$

$$= \left\{ \frac{AE}{3.25l_1} . \frac{1}{0.9\sqrt{2\pi}} .e^{\frac{-(\frac{AEv_2}{3.25l_1}-1.2)^2}{2\times 0.9^2}} \right.$$

Reliability analysis: Let us suppose now the limit displacement is $v_{21} = 5$, it is requested to find the failure probability $P_f = P(v \geq v_{21})$.

Numerical values:

$E = 5\ dan\,cm^{-2}$

$l_1 = 3$ cm

$A = 1\ cm^2$

$$\rightarrow P_f = \int_2^{\infty} PDF(v_2) = \int_2^{\infty} \frac{AE}{3.25l_1} . \frac{1}{0.9\sqrt{2\pi}} .e^{\frac{-(\frac{AEv_2}{3.25l_1}-1.2)^2}{2\times 0.9^2}} dv_2$$

$$= \int_2^{8.7562} \frac{5}{9.75} . \frac{1}{0.9\sqrt{2\pi}} .e^{\frac{-(\frac{5v_2}{9.75}-1.2)^2}{2\times 0.9^2}} dv_2 = \frac{15}{100} = 0.15$$

Comparison with monte carlo

	Proposed method	Monte carlo simulation(10000)
P_f	0.5840	0.5823

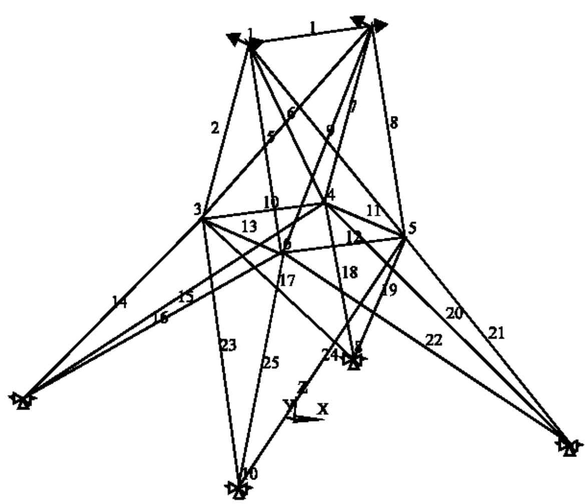


Fig. 4: 25-Bar truss structure

In the third application, we are going to analyze the reliability of a twenty five-bar truss structure (Fig. 4) with random parameters (Young modulus E, section S or the horizontal load q).

The method of the unit load permits the calculation of the displacement at a point using the following formula:

$$u = \sum_{i=1}^{n} \frac{N_i \overline{N_i}}{ES_i} L_i$$

where N_i is the normal effort due to the outside load, $\overline{N_i}$ is the normal effort due to an unit load to the point and in the direction of searching displacement, E is the Young's modulus, S_i and L_i are respectively the section and the length of the bar I.

By symmetry, the sections of some bars are identical. We adopt the following distribution:

Bar	Section
1	S_1
2,5,7,8	S_2
3,4,6,9	S_3
10,11,12,13	S_4
14,18,21,25	S_5
15,16,17,19,20,22,23,24	S_6

Normal efforts:

Bar	Vertical load	Horizontal load	Fx1 = 1	Fy1 = 1	Fz1 = 1	Fx2 = 1	Fy2 = 1	Fz2 = 1	Length Li
1	118496	1.57898E-10	-0.44778	3.7646E-17	-0.116842	0.44778	3.7646E-17	-0.116842	18000
2	-182632	-108058	0.39318	-0.88916	0.4508	0.31882	-0.04387	-0.08319	25632
3	-103094	-181168	-0.48044	-0.65356	0.101654	-0.38958	0.053606	0.101654	31321
4	-103094	24564	-0.48044	0.65356	0.101654	-0.38958	-0.053606	0.101654	31321
5	-182632	236220	0.39318	0.88916	0.4508	0.31882	0.04387	-0.08319	25632
6	-103094	-34814	0.38958	0.053606	0.101654	0.48044	-0.65356	0.101654	31321
7	-182632	-227820	-0.31882	-0.04387	-0.08319	-0.39318	-0.88916	0.4508	25632
8	-182632	99670	-0.31882	0.04387	-0.08319	-0.39318	0.88916	0.4508	25632
9	-103094	191418	0.38958	-0.053606	0.101654	0.48044	0.65356	0.101654	31321
10	-2112.2	27160	0.021874	0.075444	-0.013032	-0.021874	0.075444	-0.013032	18000
11	25438	25416	0.142222	-7.529E-17	-0.010987	0.140172	3.7646E-17	-0.071498	18000
12	-2112.2	-27160	0.021874	-0.075444	-0.013032	-0.021874	-0.075444	-0.013032	18000
13	25438	-25416	-0.140172	7.5292E-17	-0.071498	-0.142222	0	-0.010987	18000
14	-261700	-84148	0.57096	-0.52328	0.35794	0.57524	-0.6071	0.022956	32031
15	-142550	-129638	-0.147116	-0.034886	-0.041108	-0.154788	-0.54426	0.24192	43474
16	-136254	138918	0.2779	0.17921	0.17797	0.27976	0.122694	0.009951	43474
17	-142550	-78852	0.154788	-0.54426	0.24192	0.147116	-0.034886	-0.041108	43474
18	-261700	-322780	-0.57524	-0.6071	0.022956	-0.57096	-0.52328	0.35794	32031
19	-136254	-30232	-0.27976	0.122694	0.009951	-0.2779	0.17921	0.17797	43474
20	-136254	-70148	-0.27976	-0.122694	0.009951	-0.2779	-0.17921	0.17797	43474
21	-261700	116470	-0.57524	0.6071	0.022956	-0.57096	0.52328	0.35794	32031
22	-142550	133196	0.154788	0.54426	0.24192	0.147116	0.034886	-0.041108	43474
23	-136254	-38536	0.2779	-0.17921	0.17797	0.27976	-0.122694	0.009951	43474
24	-142550	75296	-0.147116	0.034886	-0.041108	-0.154788	0.54426	0.24192	43474
25	-261700	290460	0.57096	0.52328	0.35794	0.57524	0.6071	0.022956	32031

For the calculation of the horizontal displacement u_{y2} at the point 2 according the y direction, due to the load q, we put:

$$u_{y2} = \frac{q}{180000E}\left[\begin{array}{l}
\left[(1.57898e{-10})(3.7646e{-17})18000\right]\frac{1}{S_1}+ \\
\left[\begin{array}{l}(-108058)\,(-0.04387)\,25632 + (236220)\,(0.04387)\,25632 + \\ (-227820)\,(-0.88916)\,25632 + (99670)\,(0.88916)\,25632\end{array}\right]\frac{1}{S_2}+ \\
\left[\begin{array}{l}(-181168)\,(0.053606)\,31321 + (24564)\,(-0.053606)\,31321 + \\ (-34814)\,(-0.65356)\,31321 + (191418)\,(0.65356)\,31321\end{array}\right]\frac{1}{S_3}+ \\
\left[\begin{array}{l}(27160)\,(0.075444)\,18000 + (25416)\,(0)\,18000 + \\ (-27160)\,(-0.075444)\,18000 + (-25416)\,(0)\,18000\end{array}\right]\frac{1}{S_4}+ \\
\left[\begin{array}{l}(-84148)\,(-0.6071)\,32031 + (-322780)\,(-0.52328)\,32031 + \\ (116470)\,(0.52328)\,32031 + (290460)\,(0.6071)\,32031\end{array}\right]\frac{1}{S_5}+ \\
\left[\begin{array}{l}(-129638)\,(-0.54426)\,43474 + (138918)\,(0.122694)\,43474 + \\ (-78852)\,(-0.034886)\,43474 + (-30232)(0.17921)\,43474+ \\ (-70148)\,(-0.17921)\,43474 + (133196)\,(0.034886)\,43474 + \\ (-38536)\,(-0.122694)43474 + (75296)\,(0.54426)\,43474\end{array}\right]\frac{1}{S_6}
\end{array}\right]$$

After simplification, the horizontal displacement of node 2:

$$u_{y2} = \frac{q}{180000E}\left(\frac{0}{S_1} + \frac{7850940221}{S_2} + \frac{4285580903}{S_3} + \frac{73766125.44}{S_4} + \frac{1.464\,698\,375 \times 10^{10}}{S_5} + \frac{6428099237}{S_6}\right)$$

Avec:
E : Young's modulus.
q : Vertical Load.
S_i : Section of bar i.

Probabilistic study of u_{y2}: For simplification, we suppose S_i=S ➔ $u_{y2} = \frac{q}{ES}(184918.723528)$

Case 1: E uniformly distributed ➔U(10^5, 310^5)
Using our technique ➔

$$PDF(u_{y2}) = |J|\,PDF(E) = \left(\frac{q}{u^2_{y2}S}(184918.723528)\right)PDF(E)$$

$$= \begin{cases} \frac{q10^{-5}}{2u^2_{y2}S}(184918.723528) & \text{if } \frac{q(184918.723528)}{3.10^5 S} \le u_{y2} \le \frac{q(184918.723528)}{10^5 S} \\ 0 & \text{if not} \end{cases}$$

Numerical values:
q = 180000 N
S = 2000 mm^2

Reliability analysis: Let us suppose now the limit displacement is u_{y2} = 120, it is requested to find the failure probability $P_f = P(u \ge u_{y2})$.

$$➔ P_f = \int_{120}^{\infty} PDF(u_{y2})du_{y2} = \int_{120}^{\infty}\frac{q10^{-5}}{2u^2_{y2}S}(184918.723528)du_{y2} = \int_{120}^{166}\frac{16642685.118\times10^{-5}}{2.u^2_{y2}}du_{y2} = 0.19$$

Comparison with monte carlo

	Proposed method	Monte carlo simulation(10000)
P_f	0.19	0.1890

Case 2: q exponentially distributed ➔exp(1)
Using our technique ➔

$$PDF(u_{y2}) = |J|\,PDF(q) = \left(\frac{ES}{(184918.723528)}\right)PDF(q)$$

$$= \left\{\frac{ES}{(184918.723528)}.e(-\frac{ESu_{y2}}{184918.723528})\right.$$

Numerical values:
E=200000
S=2000 mm^2

Reliability analysis: Let us suppose now the limit displacement is u_{y2} = 0.0005, it is requested to find the failure probability $P_f = (u \ge u_{y2})$.

$$P_f = \int_{0.0005}^{\infty} PDF(u_{y2})du_{y2} = \int_{0.0005}^{\infty}\frac{ES}{(184918.723528)}.e(-\frac{ESu_{y2}}{184918.723528})du_{y2}$$

$$= \int_{0.0005}^{0.004} 4.610^{-4}\,e(-4.610^{-4}\,u_{y2})du_{y2} = 0.34$$

Comparison with monte carlo

	Proposed method	Monte carlo simulation(10000)
P_f	0.34	0.3382

Case 3: S normally distributed ➔N(20,1)
Using our technique ➔

$$PDF(u_{y2}) = |J|\,PDF(S) = \left(\frac{q}{u^2_{y2}E}(184918.723528)\right)PDF(S)$$

$$= \left\{\frac{q}{u^2_{y2}E}(184918.723528).\frac{1}{\sqrt{2\pi}}e^{-\frac{(\frac{q}{u_{y2}E}(184918.723528)-20)^2}{2}}\right.$$

Numerical values:
q = 180000 N
E = 200000

Reliability analysis: Let us suppose now the limit displacement is u_{y2} = 8500, it is requested to find the failure probability $P_f = P(u \geq u_{y2})$.

$$P_f = \int_{8500}^{\infty} PDF(u_{y2})du_{y2} = \int_{8500}^{10272} \frac{q}{u^2_{y2}E}(184918.723528).\frac{1}{\sqrt{2\pi}}e^{-\frac{(\frac{q}{u_{y2}E}(184918.723528)-20)^2}{2}}du_{y2}$$

$$= \int_{8500}^{10272} \frac{1.66426}{u^2_{y2}}.\frac{1}{\sqrt{2\pi}}e^{-\frac{(\frac{1.66426}{u_{y2}}-20)^2}{2}}du_{y2} = 0.33$$

Comparison with monte carlo

	Proposed method	Monte carlo simulation(10000)
P_f	0.33	0.3334

CONCLUSION

In this study, the reliability analysis of mechanical system with parameter uncertainties have been considered. The uncertainty has been considered in the material properties e.g. young modulus, cross section and in load. The method is based on the combination of the probabilistic transformation methods for a single random variable and the deterministic Finite Element Method (FEM). To proof the performance of the proposed method, the result is compared with 10000 of Monte Carlo simulation.

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