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Probabilistic Analysis of Eigenvalue of Stochastic Systems

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Abstract: The Description of real-life engineering structures systems is associated with some amount of uncertainty related to material properties, geometric parameters, boundary conditions and applied loads. In the context of structural dynamics, it is necessary to consider random eigenvalue problems in order to account for these uncertainties. A proposed approach based on the combination of the probabilistic Transformation methods for a random variable and the Rayleigh method in order to evaluate the probability density function of the eigenvalue of stochastic systems. This approach has the advantage of giving directly the whole density function (closed-form) of the eigenvalues, which is very helpful for probabilistic analysis. To show the accuracy of the proposed method, an example of a beam is analyzed for an uncertainty in the material (Young's modulus) and the geometry (beam length). The results are compared with Monte Carlo Simulations.

Key words: Random eigenvalue problem, transformation method, rayleigh method, probability density function, statistical distributions, linear stochastic systems

INTRODUCTION

The characterization of the natural frequencies and mode shapes play a fundamental role in the analysis and the design of dynamic systems. The determination of this information requires the solution of an eigenvalue problem. Eigenvalue problems also arise in the context of the stability analysis of structures. This problem could be either a differential eigenvalue problem or a matrix eigenvalue problem, depending on whether a continuous model or a discrete model is used to describe the given vibrating system. Several studies have been conducted on this topic since the mid-sixties. The study of probabilistic characterization of the eigensolutions of random matrix and differential operators is now an important research topic in the field of stochastic structural mechanics. The study by Boyce (1968) and the book by Scheidt and Purkert (1983) are useful sources of information on early work in this area of research and also provide a systematic account of different approaches to random eigenvalue problems. Several review studies, for example, by Ibrahim (1987), Benaroya and Rehak (1988), Benaroya (1992) Manohar and Ibrahim (1987-1998) and Manohar and Gupta (2003) have appeared in this field which summarize the current as well as the earlier works. The current literature on random eigenvalue arising in engineering systems is dominated by the mean-centered perturbation methods. These methods work well when the uncertainties are small and the parameter distribution is Gaussian. Methods which are not based on mean-centered perturbation but still have the generality and computational efficiency to be applicable for engineering dynamic systems are rare. In this study we obtain a closed-form expression of the probability density function of the random eigenvalue, for large uncertainties with non-Gaussian distributions.

Transformation method: Frequently, in the solution of a stochastic equation, one encounters the need to derive the probability distribution of a function of random variables. This will give the complete solution of the stochastic equation. One of the available methods for finding the distribution of a function of random variables is the Cumulative distribution function technique. That is, if X_1, X_2, X_3,..., X_n are continuous random variables with known joint PDF f (x_1, x_2, x_3,....., x_n), the distribution of a random function $Y = \mu(x_1, x_2, x_3, ..., x_n)$ is determined by computing the Cumulative Distribution Function (CDF) of Y as follows:

$$G(y) = Pr[Y \le y] = Pr[u(x_1, x_2, x_3, ..., x_n) \le y] \quad (1)$$

$$\therefore G(y) = \int_A f(x_1, x_2, x_3, ..., x_n)\, dA$$

Where A is the sub-space of points (x_1, x_2, x_3,....., x_n) in n-dimensional space defined by the inequality (x_1, x_2, x_3,....., x_n)$\le$y. Even in what superficially appears to

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be very simple, this can be quite tedious especially for irregular domain A and for complicated joint distribution function f (x_1, x_2, x_3,....., x_n). This point out the desirability of having, if possible, various methods of determination of the distribution of a function of random variables. One may find that other techniques are available but often a particular technique is superior to the others in a given situation. The soundest methods are the random variable transformation and the moment generating function techniques (Soong, 1973). In this study, we focus on the first technique. In this technique, if the joint Probability Density Function (PDF) of the input random variables X_1, X_2, X_3,..., X_n is well known in a closed form, the theory of Transformation method gives the joint probability density function of the output random functions $\{Y_i = \mu (X_1, X_2, X_3,..., X_n)\}_{i=1}^{n}$ under some mathematical conditions.

Theorem 1: Suppose that X is a continuous random variable with $f_x(x)$ and $A \subset \Re$ is the one-dimensional space where $f_x(x) > 0$. Consider the random variable (function) $Y = \mu(X)$, where $y = \mu(x)$ defines a one-to-one transformation that maps the set A onto a set $B \subset \Re$ so that the equation $y = \mu(x)$ can be uniquely solved for x in terms of y, say x = w(y). Then, the PDF of Y is:

$$g(y) = f_x\left[w(y)\right]|J|, \qquad y \in B \tag{2}$$

where,

$$J = \frac{dx}{dy} = \frac{dw}{dy}$$

is the Jacobian of the transformation, which must be continuous and not vanishing for all points $y \in B$.

Proof: Let us suppose the following two cases:

Case (1) $y = \mu(x)$ is increasing function

$$\Pr[a < Y < b] = \Pr\left[w(a) < X < w(b)\right] = \int_{w(a)}^{w(b)} PDF(x)\,dx \tag{3}$$

Changing the variable of integration from x to y where x = w(y), we obtain:

$$\Pr[a < Y < b] = \int_{w(a)}^{w(b)} PDF(x)\,dx = \int_{a}^{b} PDF\left[w(y)\right]w'(y)\,dy \tag{4}$$

From the previous integral we find:

$$g(y) = PDF\left[w(y)\right]w'(y) = PDF\left[w(y)\right]J \tag{5}$$

If we recognize $J = w'(y)$ as the reciprocal of the slope of the tangent line of the increasing function $y = u(x)$, it is the obvious that $J = |J|$ and hence,

$$g(y) = PDF\left[w(y)\right]|J| \tag{6}$$

Case (2) $y\ \mu = (x)$ is decreasing function
In this study

$$\Pr[a < Y < b] = \Pr\left[w(b) < X < w(a)\right] = \int_{w(b)}^{w(a)} PDF(x)\,dx \tag{7}$$

Changing the variable of integration, we obtain:

$$\Pr[a < Y < b] = -\int_{a}^{b} PDF\left[w(y)\right]w'(y)\,dy \tag{8}$$

From the previous integral we find:

$$g(y) = -PDF\left[w(y)\right]w'(y) = -PDF\left[w(y)\right]J \tag{9}$$

But in this study, the slope of the curve is negative and $J = -|J|$. Hence

$$g(y) = PDF\left[w(y)\right]|J|. \tag{10}$$

Rayleigh method (Boswell and Mello, 1993): The energy is supplied to a vibrating system by the applied force and the dissipation effects. During motion, the energy is stored as a kinetic energy T, a potential energy P and a strain energy V. By applying the principle of conservation of energy to an undamped system subjected to free vibration, the following relationship holds:

V+T+P = a constant

The method proposed by Rayleigh may be developed from this statement by particular reference to a single degree of freedom system.

The kinetic energy of the system is:

$$T = \frac{1}{2}m\frac{dx(t)}{dt} \tag{11}$$

While the strain energy is

$$V = \frac{k}{2}(x(t) + \delta)^2 \tag{12}$$

and the potential energy is

$$P = -mg(x(t) + \delta) \tag{13}$$

In which x(t) is the displacement from the static position (positive downward) and δ is the static displacement. Thus for vertical motion

$$\frac{1}{2}m\frac{dx(t)^2}{dt}+\frac{1}{2}k(x(t)+\delta)^2-mg(x(t)+\delta)=\text{a constant} \quad (14)$$

Rearranging and taking constants to the right-hand side

$$\frac{1}{2}m\frac{dx(t)^2}{dt}+\frac{1}{2}kx(t)^2=\text{a constant} \quad (15)$$

Assuming the displacement vary harmonically with the frequency, i.e.,

$$X(t) = A \sin wt \quad (16)$$

And substituting in previous equation, the following relationship is obtained after equating coefficients of $\sin^2 wt$

$$\frac{1}{2}mA^2w^2=\frac{1}{2}kA^2 \quad (17)$$

This equation states that maximum kinetic energy equals the maximum strain energy. These energy states occur when the system passes through the equilibrium and extreme position respectively, i.e.,

$$T_{max} = V_{max}$$

The relationship given by the previous equation can be applied to the problem of the vibration of a uniform slender beam for which the kinetic and strain energies are given by

$$T_{max}=\frac{1}{2}w^2\int_0^L my(x)^2dx$$
$$V_{max}=\frac{1}{2}\int_0^L EI\left[\frac{d^2y(x)}{dx^2}\right]^2dx \quad (18)$$

where y(x) is the amplitude of the motion or mode shape. Thus, the frequency is obtained from:

$$w^2=\frac{\int_0^L EI\left[\frac{d^2y(x)}{dt^2}\right]^2dx}{\int_0^L my(x)^2} \quad (19)$$

If the beam has concentrated masses m_1, m_2, ..., m_n at x_1, x_2, ..., x_n the frequency is obtained from

$$w^2=\frac{\int_0^L EI\left[\frac{d^2y(x)}{dt^2}\right]^2dx}{\int_0^L my(x)^2+\sum_{r=1}^{n}m_r y(x)^2} \quad (20)$$

The last equation is fundamental to the Rayleigh method in which approximations to the natural frequency are computed from assumptions regarding the mode shape. In practice the method may be used to obtain the frequency of the first mode for which suitable approximations can be made for the shape of simple structural elements.

Proposed approach:

Transformation-rayleigh method: Our proposed Approach based on the combination of Transformation and Rayleigh method. The Algorithm is in Fig. 1.

The application of this method will be demonstrated in the following example.

Application: In this application, we would like to determine the natural frequency of the fundamental mode of vibration for the beam shown in Fig. 2. The beam has a mass of m per unit length and supports a central point mass M and we suppose the Young's modulus E is random.

First step:

Applying rayleigh method: An estimate the natural frequency of the shape of the first mode of vibration is required for the beam. A reasonable choice is that shown in Fig. 2, which is the deflected shape of a mass less subjected to a central point F. the corresponding maximum deflection is assumed to be a. by applying the differential equation of flexure for the beam between A and B, which is

$$EI\frac{d^2y}{dx^2}=-\frac{Fx}{2} \quad (21)$$

And by integrating and solving for the constants, the deflected shape of the left-hand side of the beam is

$$y=a\left[3\left(\frac{x}{L}\right)-4\left(\frac{x}{L}\right)^3\right] \quad (22)$$

The maximum kinetic and strain energies of the system are now required. The maximum kinetic energy of the point mass is given by $\frac{1}{2}Mw^2A^2$ and the kinetic energy of an

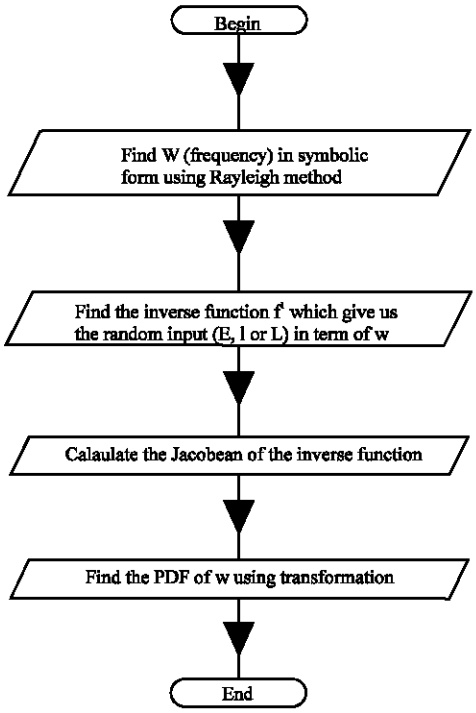

Fig. 1: Transformation-rayleigh algorithm

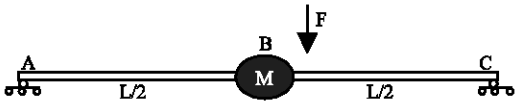

Fig. 2: beam subjected to a central point F

element dx of self mass m per unit length is $\frac{1}{2}(mdx)w^2y^2$.

integrating the latter value for the total beam and remembering that the expression for the deflection is valid between A and B, the total kinetic energy of the whole system is

$$T_{max} = 0.5Mw^2A^2 + 2\times\frac{1}{2}mw^2A^2\int_0^{L/2}\left[3(\frac{X}{L}) - 4(\frac{X}{L})^3\right]^2 dx \quad (23)$$
$$\Rightarrow T_{max} = w^2A^2(0.5M + 0.243mL)$$

Since the strain energy for the assumed deflected shape $\frac{1}{2}Fa$ is in which

$$a = \frac{FL^3}{48EI} \quad (24)$$

the maximum strain energy is

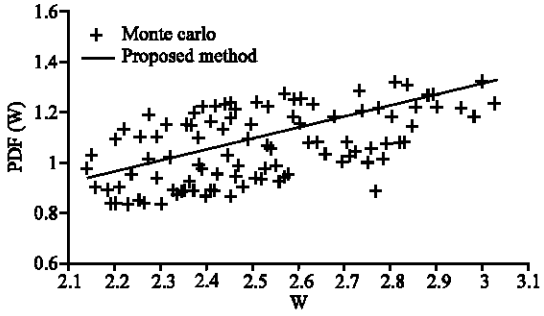

Fig. 3: Probability density function of W when E uniformly distributed

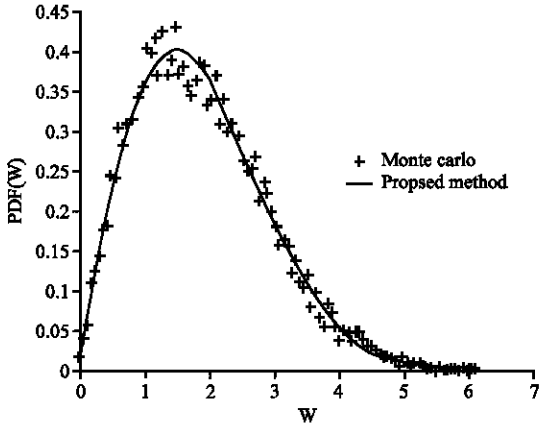

Fig. 4: Probability density function of W when E exponentially distributed

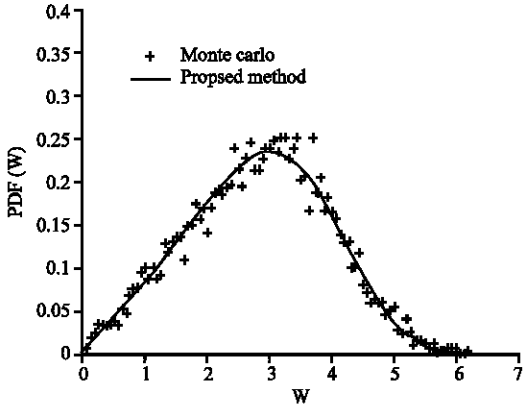

Fig. 5: probability density function of W when E normally distributed

$$V_{max} = \frac{24EIA^2}{L^3} \quad (25)$$

Therefore, from $T_{max} = V_{max}$ for the beam, the frequency is obtained from

$$w^2 = \frac{24EI}{(0.5M + 0.243mL)L^3} \quad (26)$$

Second step

Find the inverse function: From the first step, we deduce

$$w = f(E) \Rightarrow E = f^{-1}(w) \Rightarrow E = \frac{w^2(0.5M + 0.243mL)L^3}{24I} \quad (27)$$

Third step

Calculate the jacobean: From the second step,

$$|J| = \frac{w(0.5M + 0.243mL)L^3}{12I} \quad (28)$$

Fourth step

Applying the transformation method: By applying the transformation method, we obtain the probability density function of w:

$$f_w(w) = \frac{w(0.5M + 0.243mL)L^3}{12I} . f_E(E) \quad (29)$$

Case I: E random variable, uniformly distributed (E_U(1,2)) (Fig. 3)

Case II: E random variable, exponentially distributed (E_exp(1)) (Fig. 4)

Case III: E random variable, normally distributed (E_N(1,2)) (Fig. 5)

CONCLUSION

In this stuyd, the statistics of the eigenvalue of discrete linear dynamic systems with parameter uncertainties have been considered. The uncertainty has been considered in the material e.g., young modulus. Our new technique based on the combination of the transformation method and the Rayleigh method to evaluate the Probability Density Function (PDF) of the solution. Then to proof the performance of our method we compared the result with the result of 10000 simulation of Monte Carlo method.

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