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One-Stage Implicit Rational Runge-Kutta Schemes for Treatment of Discontinous Initial Value Problems

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Abstract: This study describes one-stage Implicit Rational Runge-Kutta scheme for treatment of discontinuous ordinary differential equations. Its development adopts power series expansion method (Taylor and Binomial). The analysis of its basic properties uses Dalhquist model test equation. The results show that the schemes are consistent, convergent and A-stable. Numerical computations and comparative analysis with some standard methods show that the new schemes are efficient and accurate.

Key words: One-stage implicit rational rung-kutta scheme, differential equation

INTRODUCTION

The mathematical formulation of physical situations in simulation, electrical engineering, control theory and economics often leads to an initial value problem.

$$y' = f(x, y), y(x_0) = y_0 \tag{1}$$

in which there is a pole in the solution. A simple example is the innocent looking initial value problem

$$y' = y_2,\ y(o) = 1 \tag{2}$$

whose exact solution is

$$y(x) = \frac{1}{1-x} \tag{3}$$

with a simple pole at x = 1.

ODEs possessing these type of properties are called singular/discontinuous initial value problems. Another example of such class of ODEs are IVPs in which the right-hand side function of f (2) contains discontinuities in the form of finite jumps in its components or itself or in some of its derivatives f, f', f'', $f^{(n-1)}$.

An example is the IVPs

$$y' = \begin{cases} 0, & x < 0, \\ x^6, & x \geq 0 \end{cases},\ y(-1) = 0 \tag{4}$$

whose theoretical solution is

$$y(k) = \begin{cases} 0, & x < 0, \\ x^7/7, & x \geq 0 \end{cases} \tag{5}$$

The switching on and off of electrical circuits and the state of the economy of a nation disrupted by an unforeseen circumstances or disaster are good practical examples of these problems. The development of the linear multistep methods is exclusively based on polynomial interpolation which incidentally perform very poorly in the neighbourhood of a singularity. An alternative, strategy based on non-polynomial interpolating functions was developed using either perturbed polynomials or rational functions due to pioneering effort of Ellison *et al.*[1-11]. This perhaps motivated Yuafu[22] to propose an Explicit rational Runge-Kutta Scheme of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1} V_i H_i} \tag{6}$$

where,

$$K_i = hf(x_n + c_i h_i, y_n + \sum_{j=1}^{i} a_{ij} K_j)$$

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$$H_i = hg(x_n + d_i h_i, z_n + \sum_{j=1}^{i} b_{ij} H_j) \quad (7)$$

and

$$g(x_n, z_n) = -Z_n^2 f(x_n, y_n)$$

$$Z_n = 1/y_n$$

with the constraints

$$c_i = \sum_{j=1}^{i} a_{ij}$$

$$d_i = \sum b_{ij}$$

Because of its small stability region the present work, redefined the formular as to include the implicit family of the method. That is in (6) a_{ij}# 0 for j>i, b_{ij} # 0 for j>i.

The parameters V_i, W_i, e_i, d_i, a_{ij}, b_{ij} are to be determined from the system of non-linear equations generated by adopting the following steps:

- Obtained the Taylor series of expansion of H_i's and K_i's about point (x_n, y_n) for i = 1(1)r.
- Inset the series of expansion into Eq. 8
- Compare the final expansion with the Taylor series expansion of y_{n+1} about (x_n, y_n) in the powers of h.

The numbers of parameters normally exceed the numbers of equations, but in the spirit of King, Gill and Blum, these parameters are chosen to ensure that one or more of the following conditions are satisfied.

- Adequate order of accuracy of the scheme is achieved.
- Minimum bound of local truncation error exists
- The method has maximize interval of absolute stability
- Minimum computer storage facilities.

The determination of these parameters are taking up in this unit the Scheme.

Derivation of the scheme: Setting r =1 in Eq. 6, we obtain a general one-stage implicit RR-K method of the form

$$y_{n+1} = \frac{y_n + W_1 K_1}{1 + y_n V_1 H_1} \quad (8)$$

where,

$$K_1 = (x_n + c_1 h, y_n + a_{11} k_1)$$

$$H_1 = hg(x_n + d_1 h, z_n + b_{11} H_1) \quad (9)$$

$$g(x_n, z_n) = -Z_n^2 f(x_n, y_n) \text{ and } z_n = 1/y_n \quad (10)$$

with the constraints

$$c_1 = a_{11}$$

$$d_1 = b_{11} \quad (11)$$

Adopting binomial expansion theorem on the right hand side of (11) and ignoring terms of order higher than one, yields:

$$y_{n+1} = y_n + W_1 K_1 - y_n^2 V_1 H_1 + \text{(higher order terms)} \quad (12)$$

The Taylor series of yn+1 about (xn, yn) gives

$$y_{n+1} = y_n + hy'_n - \frac{h^2}{2} y''_n + \frac{h^3}{6} y'''_n + \frac{h^4}{24} y_n^{(4)} + 0h^5 \quad (13)$$

Now.

$$y'_n = f(x_n, y_n) = f_n$$

$$y''_n = fx + f_n f_y = Df_n$$

$$y'''_n f_{xx} + 2f_n f_{xy} + f_n^2 f_{xy} + f_y (f_x + f_n f_y) = D^2 f_n + f_y Df_n$$

$$\begin{aligned} y_n^{(iv)} &= f_{xxx} + 3f_n fx_{xy} + 3f_n^2 f_{xyy} + f_n^3 f_{yyy} \\ &+ f_y (f_{xx} + 2f_n f_{xy} + f_n^2 + f_{yy}) \\ &+ (f_x + f_n f_y)(3f_{xy} + 2f_n f_y + fy^2) = \\ &D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + fy^2 Df_n \end{aligned} \quad (14)$$

where,

$$D^3 f_n = f_{xxx} + 2f_n f_{xxy} + 3f_n^2 f_{xxy} + f_n^3 f_{yyy}$$

$$D^2 f_n = f_{xx} + 2f_n f_{xxy} + f_n^2 f_{yy}$$

$$Dfy = f_{xy} + f_n f_{yy} + f_y^2$$

Substitute (17) into (16)

$$y_{n+1} = y_n + hf_n \frac{h^2}{2} Df_n + \frac{h^3}{6}(D^2 f_n + f_y Df_n) + \frac{h^4}{24}(D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + f_y^2 Df_n) + 0h^5 \quad (15)$$

Similarly expanding k_1 about (x_n, y_n)

$$K_1 = h\left(f_n + h\left(c_1 f_x + a_{11} k_1 f_y\right) + \tfrac{1}{2}\left(c_1^2 h^2 f_{xx} + 2c_1 ha_{11} k_1 f_{xy} + k_1^2 f_{yy}\right) + 0h^4 \quad (16)$$

Collecting coefficients of equal powers of h, Eq. 19 can be expressed in the form.

$$K_1 = hA_1 + h^2 B_1 + h^3 D_1 + 0h^4 \quad (17)$$

where,

$$A_1 = f_n, B_1 = C_1 (f_x + f_n f_y) = C_1 Df_n$$

$$D_1 = C_1 B_1 f_y + \tfrac{1}{2} c_1^2 (f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}) = c_1^2 (Df_n f_y + \tfrac{1}{2} D^2 f_n) \quad (18)$$

In a similar manner, expansion of H_1 about (x_n, z_n) yields

$$H_1 = hN_1 + h^2 M_1 + h^3 R_1 + 0h^4 \quad (19)$$

where,

$$N_1 = g(x_n, z_n) = g_n$$
$$M_1 = d_1 (g_x + g_n g_z) = d_1 Dg_n$$

$$R_1 = d_1 M_1 g_z + \tfrac{1}{2} d_1^2 (g_{xx} + 2g_n g_{xz} + g_n^2 g_{zz}) = d_1^2 (g_z Dg_n + \tfrac{1}{2} D^2 g_n) \quad (20)$$

with

$$Dg_n = g_x + g_n g_z$$

$$D_2 g_n = g_{xx} + 2g_n g_{xz} + g_n^2 g_{zz} \quad (21)$$

To facilitates the comparison of coefficients we can express g and its partial derivatives in terms of f and its partial derivatives.
That is, if

$$g_n = \frac{-f_n}{y_n^2}, g_x = \frac{-f_x}{y_n^2}, g_{xx} = \frac{-f_{xx}}{y_n^2}$$

$$g_z = \frac{-2f_n}{y_n} + f_y, g_{xz} = \frac{-2fx}{y_n} + f_{xy}$$

$$g_{zxz} = \frac{-2f_{xx}}{y_n} + f_{xxy}, g_{zz} = -2f_n - y_n^2 f_{yy}$$

$$g_{xzz} = -2f_x - y_n^2 f_{xyy}$$

$$g_{zzz} = 4 y_n^2 f_y + 6y_n^2 f_{yy} + 4y_n^2 f_{yyy} \quad (22)$$

Substituting (20) into (21),
We obtained

$$N_1 = \frac{-f_n}{y_n^2}, M_1 = \frac{-d_1}{y_n^2}\left(Df_n + \frac{2f_n^2}{y_n}\right)$$

$$R_1 = \frac{-d_1^2}{y_n^2}\left[\left(-\frac{2f_n}{y_n} + f_y\right)\left(Df_n + \frac{2f_n^2}{y_n}\right) + \tfrac{1}{2}\left(D^2 f_n - \frac{2f_n}{y_n}\left(\frac{f_n^2}{y_n} + f_x\right)\right)\right] \quad (23)$$

Adopting (15) and (19) in (12), we have

$$y_{n+1} = y_n + W_1 (hA_1 + h^2 B_1 + h^3 D_1 + 0h_4) - y_n^2 (V_1 (hN_1 + h^2 M_1 + h^3 R_1 + 0h^4 \quad (24)$$

$$= y_n + (W_1 A_1 - y_n^2 V_1 N_1)h + (W_1 B_1 - y_n^2 V_1 M_1)h^2 + (W_1 D_1 - y_n^2 V_1 R_1)h^3 + 0h^4 \quad (25)$$

Comparing the coefficients of the powers of h in Eq. 24 and 16, we obtained

$$W_1 A_1 - y_n^2 V_1 N_1 = f_n \quad (26)$$

From (22) and (19), we have

$$A_1 = f_n, N_1 = \frac{-f_n}{y_n^2}, (29) \text{ yields}$$

$$W_1 + V_1 = 1 \quad (27)$$

and from (22) and (19), we have

$$B_1 = C_1 Df_n, \; M_1 = \frac{d_1}{y_n^2}\left(Df_n + \frac{2f_n^2}{y_n} \right) \qquad (28)$$

Substituting (22) into (27)

$$(W_1 c_1 + V_1 d_1) Df_n + \frac{2f_n^2}{y_n} V_1 D_1 = \frac{Df_n}{2} \qquad (29)$$

From (28) we obtained

$$W_1 \; C_1 + V_1 d_1 = \tfrac{1}{2} \qquad (30)$$

Taking coefficient of h and h 2 into consideration and imposing condition

$$T_{n+1} = 0h^3 \qquad (31)$$

We obtained the following system of equations for family of one-stage of order two.

$$\begin{aligned} W_1 + V_1 &= 1 \\ W_1 C_1 + V_1 \; d_1 &= \tfrac{1}{2} \end{aligned} \qquad (32)$$

Subject to the constraints

$$\begin{aligned} a_{11} &= c_1 \\ b_{11} &= d_1 \end{aligned} \qquad (33)$$

and a local truncation error

$$T_{n+1} = (D_2 f_n + f_y Df_n)\left(\tfrac{1}{6} - \tfrac{1}{2} C_1^2 W_1 - \tfrac{1}{2} V_1 d_1^2 \left(\frac{2f_n}{y_n} - 2\frac{f_y f_n}{y_n} \right) + 0h^4 \right) \qquad (34)$$

- $W_i = 0$, $V_1 = 1$, $C_1 = d_1 = ½$, $a_{11} = b_{11} = ½$ in (8) resulting in

$$Y_{n+1} = \frac{y_n}{1 + y_n H_1} \qquad (35)$$

where,

$$H_1 = hg\,(xl + \tfrac{1}{2}h, z_n + \tfrac{1}{2}H_1) \qquad (36)$$

- $V_1 = W_1 = ½$, $C_1 = a_{11} = ¾$, $d_1 = b_{11} = ¼$ yields.

$$y_{n+1} = \frac{y_n \; \tfrac{1}{2} K_1}{1 + \frac{y_n}{2} H_1} \qquad (37)$$

where

$$\begin{aligned} Kl &= hf\,(x_n + \tfrac{3}{4}h, y_n + \tfrac{3}{4}K_1) \\ H_1 &= hg\,(x_n + \tfrac{1}{4}h, z_n + \tfrac{1}{4}H_1) \end{aligned} \qquad (38)$$

- with

$$\begin{aligned} &W_1 = 01/4, V_1 = \tfrac{3}{4}, d_1 = C_1 = \tfrac{1}{2} \\ &a_{11} = b_{11} = \tfrac{1}{2} \end{aligned}$$

yields

$$y_{n+1} = \frac{y_n + \tfrac{1}{4}K_1}{1 + 3\frac{y_n}{4} H_1} \qquad (39)$$

where,

$$\begin{aligned} K_1 &= hf\,(x_n + \tfrac{1}{2}h, \; y_n + \tfrac{1}{2}K_1) \\ H_1 &= hg\,(x_n + \tfrac{1}{2}h, \; z_n + \tfrac{1}{2}H_1) \end{aligned} \qquad (40)$$

- Case

$$W_1 = 1, V_1 = 0, C_1 = d_1 = \tfrac{1}{2}, a_{11} = b_{11} = \tfrac{1}{2}$$

In Eq. 8 yields.

$$y_{n+1} = y_n + K_1 \qquad (41)$$

where,

$$Kl = hf\,(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}K_1) \qquad (42)$$

Which incidentally coincides with Implicit Euler's Scheme of order two.

Analysis of the basic properties of the scheme: The Basic properties required of good computational methods include accuracy, consistency, convergence and stability. These properties are investigated in respect of the new schemes.

Accuracy: Characteristics of computational schemes, error can be generated when they are used to solves ODEs. The magnitude of the error determines the degree of accuracy of the schemes, if the magnitude of the error is adequately small, then the method is accurate, otherwise, It is inaccurate and their effect on solution can be terrible. It can make the solution unstable. This reason makes the analysis of the error associated with the scheme essential. The sources of these errors include local truncation error, roundoff error, discretization error and propagated error.

Round off error, is an error due to the computing device.

Mathematically, it is expressed as

$$\Upsilon_{n+1} = Y_{n+1} - P_{n+1} \qquad (43)$$

where y_{n+1} is the expected solution generated by the scheme while P_{n+1} is the computer output. However, the existing literature on error analysis by Fatunla[8] and Lambert[10] indicated that the effect of round off error can be disastrous because this will lead to inevitable loss of accuracy. This class of error is not amenable to mathematical analysis but its effect can be control by employing double precision arithmetic.

Truncation error on the other hand is the error introducedas a result of ignoring higher terms of the power series during the development of the scheme. Thus the truncation error T_{n+1} of the scheme is

$$T_{n+1} = y(x_{n+1}) - \frac{y(x_n) + W_1 K_1}{1 + y(x_n) V_1 H_1} \qquad (44)$$

where,

$$\begin{aligned} Kl &= hf\,(xl + c_1 h,\ y\,(x_n) + d_{11}\ \ K_1) \\ Hl &= hg\,(xl + d_1 h,\ z\,(x_n) + b_{11}\ \ H_1) \end{aligned} \qquad (44)$$

$$\begin{aligned} g\,(x_n, z_n) &= -Z_n^2\, f\,(x_n, y(x_n)) \\ Z_n(x_n) &=)\ \ 1\!\!\diagup\!\! y(x_n) \end{aligned} \qquad (45)$$

Assuming that the functions g and f are sufficiently differentiable in the interval of integration.

$$T_{n+1} = [D^2 f_n + f_y\, Df_n]\left(1\!/\!6 - 1\!/\!2\, C_1^2\, W_1\, 1\!/\!2\, V_1\, d_1^2 \left(\frac{2f_n}{y_n} - \frac{2f_n f_x}{y_n} \right) \right) \qquad (47)$$

Using Lotkins error bound, the bound of T_{n+1} can be found, since the bound of f and it partial derivatives defined as

$$\left| \frac{\partial^{i+j} f\,(x_n, y_n)}{\partial x^i\, \partial y^j} \right| < \frac{N^{i+j}}{M^{j-1}} \qquad (48)$$

are assumed to exist for all x $\in$ {a, b} and y $\in$ {-∞, ∞). Consequently the bound of (47) is estimated from the

$$|T_{n+1}| \le 6|P_1||N^2 M| - 4|P_2||M(N^2\!/\!2 - MN + M^2)\ \ |h^3 \qquad (49)$$

where,

$$p_1 = 1\!/\!6 - 1\!/\!2\, v_1\, d_1^2 - 1\!/\!2\, W_1\, C\, 1\!/\!2$$

$$P_2 = \frac{V_1}{y_n}\left(\frac{2f_n^2}{y_n} - \frac{2f_n f_x}{y_n} \right)$$

Discretization error en on the other hand is the difference between the exact solution y (x) at x = x_{n+1} and the numerical approximation y_{n+1}. That is

$$e_{n+1} = y_{n+1} - y\,(x_{n+1}) \qquad (50)$$

By error propagation, we mean the process by which the various errors mentioned above transferred from step to step. For instance when iterating with a numerical scheme, we obtained a sequence of approximate values y_i i =1 (1)n, if the value of y_1 has an error and y_2 depend ony1, it will inherit error from y_2, continue in this way, errors are accumulated and the final solution may be in serious inaccuracy. Such error resulting from inheritance of errors in preceeding steps is called propagated error. Such may either grow or decay.

Consistency: A numerical method is said to be consistent with the differential equation if the numerical method (11) exactly approximates the differential Eq. 1.1 to be solved Jain.

From (8) the numerical solution yn+1 at x = xn+1 is seen to satisfy the difference Eq.

$$y_{n+1} = \frac{y_n + W_1 K_1}{1 + y_n\, V_1 H_1} \qquad (51)$$

where,

$$\begin{aligned} K_1 &= hf\,(x_n + C_1\, h,\ y_n + a_{11}\ K_1) \\ Hl &= hg\,(x_n + d_1\, h,\ z_n + b_{11}\ H_1) \end{aligned} \qquad (52)$$

Subtracting y_n from both sides

$$y_{n+1} - y_n = y_n - \frac{y_n + W_1 K_1}{1 + y_n V_1 H_1} = \frac{y_n(1 + y_n V_1 H_1) - y_n + W_1 K_1}{1 + y_n V_1 H_1} \tag{53}$$

$$\frac{y_n + y_n^2 V_1 H_1 - y_n + W_1 K_1}{1 + y_n V_1 H_1}$$

$$y_{n+1} - y_n = \frac{y_n^2 V_1 H_1 + W_1 K_1}{1 + y_n V_1 H_1} \tag{54}$$

$$(y_{n+1} - y_n) = \frac{y_n^2 V_1 H_1}{1 + y_n V_1 H_1} + \frac{W_1 K_1}{1 + y_n V_1 H_1} \tag{55}$$

But by definition

$$\begin{aligned} K_1 &= hf(x_n + C_1 h, y_n + a_{11} K_1) \\ H_1 &= hg(x_n + d_1 h, z_n + b_{11} H_1) \end{aligned} \tag{56}$$

Using this in Eq. 55 and divide both sides of (54) by h and take limit as h tends to zero.

$$\frac{y_{n+1} - y_n}{h} = \left(\frac{y_n^2 V_1 H_1 + W_1 K_1}{1 + y_n V_1 H_1}\right)\frac{1}{h}$$

$$= \frac{\dfrac{y_n^2 V_1 hg(x_n + d_1 h, Z_n + b_{11} H_1) + W_1 hf(x_n + c_1 h, y_n + a_{11} K_1)}{1 + y_n V_1 hg(x_n + d_1 h, Z_n + b_{11} H_1)}}{h}$$

$$\lim_{h \to o} \frac{y_{n+1} - y_n}{h} = \lim_{h \to o} - \frac{\dfrac{y_n^2 V_1 hg(x_n + d_1 h, Z_n + b_{11} H_1) + W_1 hf(x_n + c_1 h, y_n + a_{11} K_1)}{1 + y_n V_1 hg(x_n + d_1 h, Z_n + b_{11} H_1)}}{h}$$

$$\lim_{h \to o} \frac{y_{n+1-y_n}}{h} = \frac{y_n^2 V_1 f(x_n y_n) + W_1 f(x_n, y_n)}{1 + y_n . Vf(x_n, y_{n1})}$$

That is;

$$y'_n = f(x_n, y_n)$$

Hence the method is consistent.

Convergence: Adopting the numerical scheme (8) for solving the IVPs in (1) the method will be convergent, if the numerical approximation yn+1 generated by it tends to the exact solution y (x_{n+1}) of the IVPs as the step size tends to zero.

That is,

$$\lim_{h \to \infty} [y(x_{n+1}) - y_{n+1}] = 0 \tag{57}$$

This property is investigated for the scheme here.

Let e_{n+1} and T_{n+1} denote the discretization error and truncation errors generated by (8), respectively. Adopting binomial expansion and ignoring higher terms in Eq. 8 and 44 we obtain

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + h\Psi_2(x_n, y(x_n); h) + \\ & h\phi_1(x_n, y(x_n); h) + (\text{higherterms}) + T1 \end{aligned} \tag{58}$$

where ϕ_1 , ψ_2 and ϕ_2 are continous functions in the domain

$$a \le x \le b, |y| < \infty, 0 \le h \le h_o$$

$$\text{Let } h\phi_1(x_n, y(x_n); h) = W_1 K_1 \tag{59}$$

$$h\phi_2(x_n, z(x_n); h) = V_1 H_1 = \frac{h}{y^2(x_n)} \psi_2(x_n, y(x_n); h) \tag{60}$$

where

$$\psi_2 (x_n, y(x_n); h) = (1 + y(x_n) b_{11} H_1) \phi_2(x_n, y(x_n); h) \tag{61}$$

Similarly Eq. 8 yields

$$y_{n+1} = h\psi_2 (x_n, y_n; h) + h\psi_1(x_n, y_n; h) + \text{higher terms} \tag{62}$$

Subtract Eq. 58 from 62 and use Eq. 50 leads to

$$\begin{aligned} e_{n+1} &= e_n + h[\psi_2(x_n, y(x_n); h) - (x_n, y_n; h)] + \\ & h\phi_1(x_n y(x_n); h) - \phi_1(x_n, y_n; h)] \end{aligned} \tag{63}$$

By taking the absolute value on both sides of Eq. 63, we have the inequality.

$$|e_{n+1}| \le |e_n| + Kh|e_n| + hL|e_n| + T \tag{64}$$

Where L and K are the Lipschitz constants of the functions.

ϕ_1 (x, y; h) and ψ_2(x, y; h), respectively

Let $T = \sup_{a \le x \le b} |T_{n+1}|$ (65)

By setting N = L + K (66)

Inequality (64) becomes

$$|e_{n+1}| \le |e_n|(1+hN)+T, n=0,1,2 \quad (67)$$

From theorem in Lambert[6] quoted without proof states that if $\{e_j, j = o\ (1)\ n\}$ be sets of real number. Such that the exist finite constants R and S

$$|e_j| \le R\ |e_{j-1}| + S, j=o(1)n-1 \quad (68)$$

$$r_j \le \left(\frac{R^j - 1}{R-1}\right) S + R^j |e_o|, R \ne 1 \quad (69)$$

Adopting this theorem for the inequality (67), we have

$$|e_n| \le \frac{(1+hN)^n - 1)}{hN} + (1+hN)^n\ |e_o| \quad (70)$$

Since $(1 + hN)^n$ can be approximated by

$$(1+hN)^n = e^{nhN} = e^{N(x_n)}$$

Where x_n = nh, n = o (1)N and $x_n \le$ b then $x_n - a \le b - a$. Consequently,

$$e^{N(x_n)} \le e^{N(b-a)} \quad (71)$$

$$e_n \le \left(\frac{e^{N(b-a)} - 1}{hN}\right) T + e^{N(b-a)} |e_o| \quad (72)$$

Considering Eq 58 and adopting first mean value theorem

$$\begin{aligned} T_{n+1} &= h[\psi(x_n+\theta h, y(x_n+\theta h)) - \psi_2(x_n, y(x_n)) \\ &+ h[\phi_1(x_n+\theta h, y(x_n+\theta h)) - \phi_1(x_n; y(x_n))] \\ &= h\begin{bmatrix} \psi_2(x_n+\theta h, y(x_n+\theta h)) - \psi_2(x_n+\theta h, y(x_n)) \\ +\psi_2(x_n+\theta h, y(x_n)) - \psi_2(x_n, y(x_n)) \\ +h\phi_1(x_n+\theta h, y(x_n+\theta h)) - \phi_1(x_n+\theta h, y(x_n)) \\ +\phi_1(x_n+\theta h, t(x_n)) - \phi_1(x_n, y(x_n)) \end{bmatrix}; 0 \le \theta \le 1 \end{aligned} \quad (73)$$

By taking the absolute values of (73) into consideration, we have

$$T = hL|y(x_n+\theta h) - y(x_n)| + jh^2\theta + hK|y(x_n) + \theta h - y(x_n)| + Mh^2\theta \quad (74)$$

$$T = h^2\theta N y'(\in_1) + (J+M)h^2\theta, x_n \le \in_1 \le x_{n+1} \quad (75)$$

Where M and J are the partial derivatives of ϕ_1 and ψ_2 with respect to x, respectively.

By setting Q = J + M (76)

and

$$Y = \sup_{a \le x \le b} Y'(x) \quad (77)$$

Therefore, Eq. 75 yield

$$T = h^2\theta\ (NY + Q) \quad (78)$$

By substituting (18) into (72),we have

$$|e_n| \le h\theta e^{N(b-a)}(NY+Q) + e^{N(b-a)}|e_o| \quad (79)$$

Assuming no error in the input data, that is, $e_o = 0$, then in the limit as h tends to zero, we obtain

$$\lim_{\substack{h\to 0 \\ h\to\infty}} |e_n| = 0 \quad (80)$$

which implies from 50 that

$$\lim_{\substack{h\to 0 \\ h\to\infty}} Y_n = y(x_n) \quad (81)$$

Stability properties: As mentioned earlier that any error introduced at any stage of the computation which is not bounded can produced unstable numerical results. Therefore, we consider the stability analysis of the proposed one-stage Implicit Rational R-K schemes in (11) to access its suitability.

To achieve this, we apply scheme (11) to stability scalar test initial value problem.

$$Y' = \lambda y, y(x_o) = y_o \quad (82)$$

The general one-stage scheme is of this form

$$y_{n+1} = \frac{y_n + W_1 K_1}{1 + y_n V_1 H_1} \quad (83)$$

where,

$$K_1 = hf(x_n + C_1 h_1 \; y_n + a_{11} K_1)$$
$$Hl = hg(x_n + d_1 h_1 \; Z_n + b_{11} H_1)$$

$$g(x_n, z_n) = Z_n^2 f(x_n, y_n) \quad (84)$$

applying (83) to the stability test Eq. 82,we obtained the recurrent relation

$$y_{n+1} = \left(\frac{1 + W_1^T (1 - a_{11}\hbar)^{-1}}{1 - V_1^T (1 + b_{11}\hbar)^{-1}} \right) y_n \quad (85)$$

By setting

$$\mu(\hbar) = \frac{1 + W_1^T (1 - a_{11}\hbar)^{-1}}{1 - V_1^T (1 + b_{11}\hbar)^{-1}} \quad (86)$$

For example, the associated stability function for scheme (72) is

$$\mu(\hbar) = \frac{1 + \frac{1}{2}\hbar}{1 - \frac{1}{2}\hbar} \quad (87)$$

which is (1) Pade's approximation to e^h since

$$\mu(\hbar) = 1 + \hbar + \frac{1}{4}\hbar^2 + 0\hbar^3 \quad (347)$$

Also the numerical scheme (8) is convergent. If $|\mu(\hbar)| < |$ analysis (87), we have that its interval of A-stability is (-8, 0). This implies that the scheme is A-stable.

Computer implementation and numerical computation: In order to test for the performance of this scheme, we computerised the formulas and implement them on a microcomputer using sample problems and compare the results with the conventional Runge-Kutta scheme of the same order. of the form

$$y_{n+1} = y_n + hk_1 \quad (89)$$

where,

$$K_1 = hf(x_n + h, y_n + k_1) \quad (90)$$

Problem 1: Consider problem

$$y^1 = y^2, \; y(o) = 1, \; 0 < x \le 1 \quad (91)$$

whose theoretical solution is

$$y = \frac{1}{1-X} \quad (92)$$

The results are as shown in Table 1.

Problem 2: The initial value problem

$$y^1 = 10\,(y - 1)^2, \; y(o) = 1 \quad (93)$$

with exact solution

$$Y(x) = 1 + \frac{1}{1 + 10x} \quad (94)$$

The results are as shown in Table 2.

General discussion on the results of the treatment of one-stage implicit rational R-K schemes: This section discusses the outcome of the application of the

Table 1: Numerical results of problem one with one stage implicit rational runge- kutta schemes of order one and one stage classical r- k scheme of order one

XN	Yexact	One stage implicit rational R-K scheme	One stage implicit rational R-K scheme error	One stage classical R-K scheme	One stage classical R-K scheme error
0.10000000D + 00	0.150000000D + 01	0.150000000D + 01	0.5000000D-06	0.14900000D + 01	0.49000000D-06
0.20000000D + 00	0.13333330D + 01	0.13000000D + 01	0.33333334D-06	0.12900000D + 01	0.33293340D-06
0.30000000D + 00	0.12500000D + 01	0.12500000D + 01	0.25000000D-04	0.12500000D + 01	0.24900000D-04
0.40000000D + 00	0.12000000D + 01	0.12000000D + 01	0.20000000D-05	0.12000000D + 01	0.20000000D-05
0.50000000D + 00	0.11666670D + 01	0.11500000D + 01	0.16666660D-06	0.11500000D + 01	0.16666659D-06
0.60000000D + 00	0.11428570D + 01	0.11390000D + 01	0.14285720D-05	0.11380000D + 01	0.14285720D-05
0.70000000D + 00	0.11250000D + 01	0.11249000D + 01	0.1250000D-04	0.11123000D + 01	0.11500000D-04
0.80000000D + 00	0.11111110D + 01	0.11100000D + 01	0.11111100D-05	0.11100000D + 01	0.11111100D-05
0.90000000D + 00	0.11200000D + 01	0.11200000D + 01	0.10000000D-04	0.11120000D + 01	0.10000000D-04
0.10000000D + 01	0.10800000D + 01	0.10800000D + 01	0.90909120D-06	0.10700000D + 01	0.90909120D-06

Table 2: Numerical results of problem two with one stage implicit rational runge- kutta schemes of order two and two stage classical r- k scheme of order two

XN	Yexact	One stage implicit rational R-K scheme	One stage implicit rational R-K scheme error	One stage classical R-K scheme	One stage classical R-K scheme error
0.10000000D + 00	0.26887940D + 01	0.26675100D + 01	0.32986300D-03	0.27093440D + 01	0.23404650D-03
0.20000000D + 00	0.71173530D + 01	0.71179980D + 01	0.79746630D-04	0.72297450D + 01	0.70864100D-04
0.30000000D + 00	0.18987060D + 02	0.18985990D + 02	0.88909200D-04	0.19283130D + 02	0.19206340D-04
0.40000000D + 00	0.506256401D + 02	0.50621320D + 02	0.40539000D-05	0.51411080D + 02	0.51328770D-05
0.50000000D + 00	0.13473790D + 03	0.13494900D + 03	0.33481630D-05	0.13705080D + 03	0.13691910D-05
0.60000000D + 00	0.35977880D + 03	0.35975450D + 03	0.45953600D-05	0.36536130D + 03	0.36514290D-05
0.70000000D + 00	0.95951190D + 03	0.95913450D + 03	0.75879060D-06	0.97409270D + 03	0.97374880D-06
0.80000000D + 00	0.25573560D + 04	0.25573300D + 04	0.35568170D-06	0.25972390D + 04	0.25967260D-06
0.90000000D + 00	0.68189370D + 04	0.68189730D + 04	0.48182440D-04	0.69254430D + 04	0.69247140D-04
10000000D + 00	0.18183460D + 04	0.18183080D + 04	0.38182080D-05	0.18467080D + 05	0.18466080D-05

proposed scheme in solving the sample problems 1 and 2. Table 1 and 2 show that the proposed scheme compete favourably with the existing Implicit Classical Runge-Kutta method of order 2. Moreso, one can deduce from the results in the Table 1 and 2 that the result converge and accurate. Also numerical computations and comparative analysis with some standard methods show that the new scheme is efficient and accurate.

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