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Function Approximation by Feed Forward Neural Networks with a Fixed Weights Using Sigmoidal Signals

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Abstract: Neural networks have been successfully applied to various pattern recognition and function approximation problems. The author recently introduced left sigmoidal signals and right sigmoidal signals to prove certain function approximation theorems for feed forward neural networks. In this study, by imposing certain conditions on the continuous functions on R, we find those conditions that can be approximated by feed forward neural networks with fixed weights using left sigmoidal signals and right sigmoidal signals.

Key words: Feed forward networks, approximation, sigmoidal signals, activation functions etc

INTRODUCTION

The approximation capabilities of multi-layer neural networks have been investigated by many investigators[1-4]. It has been proved that three layered neural networks with sigmoidal activation functions can approximate a set of continuous functions which is defined in $\mathbb{R}^n$ and for which $\lim_{|x|\to\infty} f(x)$ exists to any desired degree of accuracy[5]. Arbitrary continuous functions over are not generally approximated by Neural Networks with sigmoidal activation function.

For instance, f(x) = x is continuous on R, but it cannot be approximated by a Neural Network with sigmoidal functions over with respect to suprimum norm as $|f(x)| \to \infty$ a $x \to \infty$. However, Hong and Nahm[6] established that continuous functions f on R with $\lim_{|x|\to\infty} f(x) = 0$, can be approximated by a bounded sigmoidal signal over R.

The authors introduced left sigmoidal and right sigmoidal signals to approximate bounded continuous functions for Feed Forward Neural Networks with one hidden layer but not all continuous functions defined on R can be approximated by the left and right sigmoidal signals.

For example

$$f(x) = \begin{Bmatrix} x & x \le 0 \\ 0 & x > 0 \end{Bmatrix}$$

and $$g(x) = \begin{Bmatrix} 0 & x < 0 \\ x & x \ge 0 \end{Bmatrix}$$

are not approximated by sigmoidal signals (left or right) even though they are continuous on R. In this study, by imposing certain conditions on the continuous functions on, we find those continuous functions that can be approximated by left (or) right sigmoidal signals.

PRELIMINARIES

Definition 2.1 σ: R→R is said to be a generalized sigmoidal function

$$\text{if } \lim_{|x|\to\infty} \sigma(x) = 0 \text{ and } \lim_{|x|\to\infty} \sigma(x) = 1$$

Theorem 2.2: Let be a bounded sigmoidal function on R and let $\epsilon > 0$ be given. If f is a continuous function on R such that $\lim_{|x|\to\infty} f(x) = 0$, then there exist constants $b_i, c_i \in$ R and positive integers k, N such that

$$\left| f(x) - \sum_{i=1}^{N} c_i\, \sigma(kx + b_i) \right| < \varepsilon, \quad x \in \mathbb{R}$$

Definition 2.3: The function σ: R→R is said to be left sigmoidal if $\lim_{|x|\to\infty} \sigma(x) = 0$.

The function σ: R→R is said to be right sigmoidal if $\lim_{|x|\to\infty} \sigma(x) = 1$.

Function approximation with fixed weights

Theorem 3.1: Let σ be a bounded right sigmoidal function on R and $\epsilon > 0$ be given. If f(x) is a continuous function

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on R such that $\lim_{|x|\to\infty} \sigma(x) = 0$, then there exist constants $b_i, c_i, \in R$ and positive integers k, N such that

$$\left| f(x) - \sum_{i=1}^{N} c_i \ \sigma(kx + b_i) \right| < \varepsilon, \forall\ x \in (0, \infty)$$

Proof: Since f is uniformly continuous on $[0,\infty]$, there exists $f > 0$ such that $|f(x) - f(y)| < \epsilon/8\|\sigma\|$ for any x, y $\in [0,\infty]$ with, $|x-y| < \delta$, then $\|\sigma\| > 1$.

Hence,

$\lim_{|x|\to\infty} f(x) = 0$, there exists a positive integer L such that $|f(x)| < \epsilon/8\|\sigma\|$ for $x \geq L$.

Let N = max $\{L, [1/\delta]+1\}$, where $[\bullet]$ is the Gauss function.

We divide the interval [0, N] into $2N^2$ equal intervals by means of point x_i, where $0 = x_0 < x_1 < \ldots < x_i < \ldots < x_{2N^2} = N$.

Define $b_i = (x_i + x_{i+1})|2$ for $0 \leq I \leq 2N^2-1$ since σ is a right sigmoidal function, there exists $r > o$ such that $|\sigma(x) - 1| < 1/N^2$ for $x \geq r$.

Choose a positive integer k such that $k/2N > r$.

Now we construct a network

$$g(x) = \sum_{i=1}^{2N^2} \left(f(x_i) - f(x_{i-1}) \right) \ \sigma(k(x - b_i))$$

If $x \in [0,N]$, then $x \in [x_{i0-1}, x_{i0}]$ for some i_0 with $1 \leq i_0 \leq 2N^2$.

If $x \geq N$, then $k(x - b_i) \geq r$ and hence $|\sigma(k(x-bi))\ 1| \leq 1/N^2$ for i = 1, 2,… ,$2N^2$.

Then

$$|f(x) - g(x)| = \left| f(x) - \sum_{i=1}^{2N^2} ((f(x_i) - f(x_{i-1}))\right.$$

$$\left. (\sigma(k(x - b_i)) - 1) + \sum_{i=1}^{2N^2} (f(x_i) - f(x_{i-1})) \right|$$

$$= \begin{array}{l} |f(x)| + |f(N)| + |f(-N)| + \\ \sum_{i=1}^{2N^2} |f(x_i) - f(x_{i-1})| \ (\sigma(k(x - b_i)) - 1) \end{array}$$

$$< \frac{\varepsilon}{8\|\sigma\|} + \frac{\varepsilon}{8\|\sigma\|} + \frac{\varepsilon}{8\|\sigma\|} + \frac{2\varepsilon}{8\|\sigma\|}$$

$<\epsilon$.

Theorem 3.2: Let $\sigma(x)$ be a bounded left sigmoidal function on R and f is a continuous function on such that $\lim_{|x|\to\infty} f(x) = 0$. Then there exist constants b_i, c_i $\in R$ and positive integer k, N such that

$$\left| f(x) - \sum_{i=1}^{N} c_i \, \sigma\left(kx + b_i\right) \right| < \varepsilon, \forall\ x \in (-\infty, 0)$$

Proof: Let $\|\sigma\| = \sup |\sigma(x)|$, $-\infty < x < 0$. Since f is uniformly continuous on R, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/8\|\sigma\|$ for any x, y $\in(-\infty, 0)$ with $|x-y| < \delta$, then $\|\sigma\| \geq 1$.

Since $\lim_{|x|\to\infty} f(x) = 0$, there exists an integer L such that $|f(x)| < \epsilon/8\|\sigma\|$ for $x \leq L$.

Let N = max $\{L, [1/\delta + 1\}$, where $[\bullet]$ is the Gauss function.

Now we divide the interval [-N, 0] into $2N^2$ equal intervals

where $-N = x_0 < x_1 < \ldots < x_{2N^2} = 0$. Define $b_i = (x_i + x_{i+1})/2$ for $0 \leq i \leq 2N^2-1$.

Since σ is a left sigmoidal function, there exists $r \in R$ such that $|\sigma(x)| < 1/N^2$ for $x \leq -r$. Choose a positive integer k such that $k/2N > r$.

Now construct a network

$$g(x) = \sum_{i=1}^{2N^2} \left(f(x_i) - f(x_{i-1}) \right) \ \sigma(k(x - b_i))$$

If $x \leq -N$, then $k(x-b_i) \leq -r$ and hence, $|\sigma(k(x-b_i))| \leq 1/N^2$ for $i = 1, \ldots, 2N^2$

$$\therefore |f(x) - g(x)| \leq |f(x)| + |g(x)|$$

$$\leq \frac{\varepsilon}{8\|\sigma\|} + \sum_{i=1}^{2N^2} |f(x_i) - f(x_{i-1})| \ \frac{1}{N^2}$$

$$\leq \frac{\varepsilon}{8\|\sigma\|} + \frac{2\varepsilon}{8\|\sigma\|}$$

$<\epsilon$.

If $x \in [-N, 0]$, then $x \in [x_{i-1}, x_i]$, for some i_o with $1 \leq i_o \leq 2N^2$.

Note that $k(x - b_i)$ r for $i = 1, \ldots, i_{o-1}$ and $k(x - b_i) \leq -r$ for $i = i_{o+1}, \ldots, 2N^2$.

From the fact that

$$\sum_{i=1}^{i_0-1} \left(f(x_i) - f(x_{i-1}) \right) \ \sigma\left(k(x - b_i)\right)$$

$$= \sum_{i=1}^{i_0-1} \left(f(x_i) - f(x_{i-1}) \right) \left(\sigma\left(k(x - b_i)\right) - 1 \right) - f(x_{i_o - 1}) + f(x_o)$$

$\therefore$ We have

$$|f(x) - g(x)| = \left| f(x) - \sum_{i=1}^{i_0-1} \left(f(x_i) - f(x_{i-1}) \right) \ \sigma\left(k(x - b_i)\right) \right|$$

$$+ \left(f(x_{i_0+1}) - f(x_{i_0}) \right) \ \sigma\left(k\left(x - b_{i_0}\right)\right)$$

$$+\sum_{i=i_0+1}^{2N^2}\left(f(x_i) - f(x_{i-1})\right)\ \sigma\left(k(x-b_i)\right)$$

$$\le \left|f(x) - f(x_{i_0-1})\right| + f(-N) + \sum_{i=1}^{i_0-1}\left|f(x_i)-f(x_{i-1})\right|\ \left|\sigma\left(k\left(x-b_i\right)-1\right)\ \right|$$

$$+\left|f(x_{i_0+1})-f(x_{i_0})\right|\left|\sigma(k(x-b_{i_o}))\right| + \sum_{i=i_0+1}^{2N^2}\left|f(x_i)-f(x_{i-1})\right|\left|\sigma(k(x-b_i)\right|$$

$$\le \frac{\varepsilon}{8\|\sigma\|} + \frac{\varepsilon}{8\|\sigma\|} + \frac{2\varepsilon}{8\|\sigma\|} + \frac{\varepsilon}{8\|\sigma\|}\|\sigma\| + \frac{2\varepsilon}{8\|\sigma\|}$$

$<\epsilon$

Definition 3.3: Let f and g be any two functions on R. The convolution of f and g Is defined by (f*g) (x) $= \int_{\mathbb{R}} f(y)\ \ g(x-y)\ \ dy$.

For any x ϵ R, We define $HL(x) = \begin{cases} c\ \ e^{\frac{-1}{x}} & ,0<x<1 \\ 0 & \text{otherwise} \end{cases}$

where c is chosen so that $\int HL(x)\ \ dx = 1$ $\int_{\mathbb{R}} HL(x)\ \ dx = 1$

Then, HLϵC^{∞}_c.

For each positive integer k, we define $HL_k(x) = k\ HL\ (k\ x)$.

Then $\int_{\mathbb{R}} HL_k(x)\ \ dx = 1$ and Hl_k ε ϵC^{∞}_c for any positive integer k.

For any xϵ R, we define

$$HR(x) = \begin{cases} c\ \ e^{-\frac{1}{x}} & ,-1<x<0 \\ 0 & \text{otherwise} \end{cases}$$

where c is chosen so that Then HR ϵC^{∞}_c .

For each positive integer k, we define $HR_k(x) = k\ HR\ (kx)$.

Then $\int_{\mathbb{R}} HL_k(x)\ \ dx = 1$ and HR_k for any positive integer k.

Lemma 3.4: If f is a uniformly continuous function on R, then HL_k* f converges uniformly to f on R.

Proof : Let ε > 0 be given.

Then, there exists a positive integer N such that |f(x)- f(y)| <ϵ

for any x, y ϵ R with |x-y| < 1/N, since f is uniformly continuous.

For any x ϵ R and any integer k with k ≥ N, we have

$$\left|\left(HL_k *\ \ f\right)(x)\ \ -f(x)\right|$$

$$= \left|\int_{\mathbb{R}} k\,HL(ky)\left(f(x-y) - f(x)\right)\ dy\right|$$

$$\le \int_{\mathbb{R}}\left|HL(z)\right|\ \left|f(x-\frac{z}{k}) - f(x)\right|\ dz$$

where z = ky

$$\le \varepsilon \int_0^1 \left|HL(z)\right|\ dz$$

= ϵ

This shows that HL_k * f converges uniformly to f on R.

Lemma 3.5: If f is uniformly continuous function R, then HR_k * f converges uniformly to f on R.

Proof: Let ε > 0 be given. Then, there exists a positive integer N such that |f(x) - f(y)| <ε for any x, y ϵ R with |x-y| < 1/N, since f is uniformly continuous.

For any x ϵ R and any integer k with k ≥ N, we have

$$\left|\left(HR_k * f\right)(x) - f(x)\right| = \left|\int_{\mathbb{R}} k\,HR(ky)\ \ (f(x-y) - f(x))\ \ dy\right|$$

$$= \le \int_{\mathbb{R}}\left|HR(z)\right|\ \left|f(x-\frac{z}{k}) - f(x)\right|\ dz$$

$$= \int_{-1}^{0}\left|HR(z)\right|\ \left|f\left(x-\frac{z}{k}\right) - f(x)\right|\ dz$$

$$\le \varepsilon \int_{-1}^{0}\left|HR(z)\right|\ dz$$

= ϵ

This shows that HR_k * f converges uniformly to f on R.

Theorem 3.6: Let f be a continuous function on a bounded closed interval [-a, 0] of R. If σ is a bounded measurable left sigmoidal function on R, then there exist constants b_i, c_i ϵ R and positive integers k, N such that

$$\left| f(x) - \sum_{i=1}^{N} c_i \sigma(kx + b_i) \right| < \varepsilon, \quad -a \le x \le 0$$

Proof : We construct a uniformly continuous function $\bar{f}$ on R such that

$\bar{f} = f$ on [-a, 0] and $\bar{f} = 0$ outside of [-a-1, 1].

By lemma, $HL_k * \bar{f}$ uniformly converges to $\bar{f}$ on and hence, $HL_k * \bar{f}$ uniformly converges to f on [-a, 0].
Since $\int Hl_k(x-y)\ (y)\bar{f}\, dy < \infty$ for each positive integer k, the convolution
$HL_K * \bar{f}$ is approximated by a Riemann sum.

For each positive integer k, there exist a positive integer M_k and constants y_i, c_i for i =1,…, M_k such that

$$\left| \left(HL_K * \bar{f}\right)(x) - \sum_{i=1}^{M_K} c_i\, HL_k(x - y_i)\bar{f}(y_i) \right| < \frac{\varepsilon}{3}, \qquad (1)$$

where $y_i \in R$ for i = 1,…, M_k

Since $HL_k \in Co$ where Co denotes the collection of all continuous functions that covers to 0 as |x| approaches to ∞, by theorem 3.1, there exist constants $\alpha_{j,k}$, $\beta_{j,k} \in R$ and a positive integer L such that

$$\left| HL_K(x - y_i) - \sum_{j,k} \beta_{j,k}\sigma(L(x - y_i) + \alpha_{j,k}) \right| < \frac{\varepsilon}{3} \qquad (2)$$

Choose a positive integer k such that

$$|f(x) - (Hl_k * \bar{f})(x)| < \varepsilon/3, \text{ for } x \qquad (3)$$

From (1) – (3), we get

$$\left| f(x) - \sum_{i=1}^{M_K} c_i \bar{f}(y_i) \sum_{j,k} \beta_{j,k}\sigma(L(x - y_i) + \alpha_{j,k}) \right|$$

$$\le \left| f(x) - (HL_k * \bar{f})(x) \right| +$$
$$\left| (HL_k * \bar{f})(x) - \sum_{i=1}^{M_k} c_i\, HL_k(x - y_i)\bar{f}(y_i) \right| +$$
$$\left| \sum_{i=1}^{M_K} c_i HL_K(x - y_i)\bar{f}(y_i) - \sum_{i=1}^{M_K} c_i \bar{f}(y_i) . \sum_{j,k} \beta_{j,k}\sigma(L(x - y_i) + \alpha_{j,k}) \right|$$
$$< \varepsilon$$

This completes the proof.

Theorem 3.7: Let f be a continuous function on a bounded closed interval [o, b] of R. If σ is a bounded measurable right sigmoidal function R, then there exist constants $b_i, c_i \in R$ and positive integers k, N such that

$$\left| f(x) - \sum_{i=1}^{N} c_i \sigma(kx + b_i) \right| < \varepsilon, \quad 0 \le x \le b$$

Proof: we construct a uniformly continuous function $\bar{f}$ on R such that $\bar{f} = f$ on [o, b] and $\bar{f} = 0$ outside of [-1, b+1]. By Lemma $HR_K * \bar{f}$ uniformly converges to on and Hence $HR_K * \bar{f}$ uniformly converges to f on [o, b]. Since $\int HR_k(x-y)\ (y)\bar{f}\, dy < \infty$ foreach positive integer k, the convolution $HR_K * \bar{f}$ is approximated by a Riemann sum.

For each positive integer k, there exist a positive integer M_k and constant y, c_i for i=1,…, M_k such that

$$\left| (HR_K * \bar{f})(x) - \sum_{i=1}^{M_K} c_i (HR_K(x - y_i)\bar{f}(y_i) \right| < \frac{\varepsilon}{3} \qquad (4)$$

where $y_i \in R$ for i=1,……, M_k

Since $HR_K \in Co$, by theorem 3.2, there exist constant $\alpha_{j,k}$, $\beta_{j,k} \in R$ and a positive integer L such that

$$\left| HR_K(x - y_i) - \sum_{j,k} \beta_{j,k}\sigma(L(x - y_i) + \alpha_{j,k}) \right| < \varepsilon/3 \qquad (5)$$

Choose a positive integer k such that

$$\left| f(x) - (HR_K * \bar{f})(x) \right| < \varepsilon/3, \text{for } x \in \mathbb{R} \qquad (6)$$

From (4) to (6), we get

$$\left| f(x) - \sum_{i=1}^{M_K} c_i \bar{f}(y_i) \sum_{j,k} \beta_{j,k}\sigma(L(x - y_i) + \alpha_{j,k}) \right|$$
$$\le \left| f(x) - (HR_K * \bar{f})(x) \right|$$
$$+ \left| (HR_K * \bar{f})(x) - \sum_{i=1}^{M_K} c_i\ HR_K(x - y_i)\bar{f}(y_i) \right|$$
$$+ \left| \sum_{i=1}^{M_K} c_i HR_K(x - y_i)\bar{f}(y_i) - \sum_{i=1}^{M_K} c_i \bar{f}(y_i) \sum_{j,k} \beta_{j,k}\sigma(L(x - y_i) + \alpha_{j,k}) \right|$$
$$< \varepsilon$$

This completes the proof.

CONCLUSION

Thus we have introduced left sigmoidal signals and right sigmoidal signals that have the function approximation[7]. We have proved that arbitrary continuous functions in R, with fixed weights in (-∞, 0) and (0,∞) can be approximated by bounded left sigmoidal signals and right sigmoidal signals. We used the left sigmoidal signals and right sigmoidal with fixed weights to prove certain function approximation theorem due to Nahm.

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