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Triangles in Lattice Parabola and Lattice Cubic

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Abstract: This study gives some properties in connection with the triangles inscribed in the lattice parabola $y = x^2$ and the lattice cubic $y = x^3$.

Key words: Triangles, Lattice Parabola, Lattice Cubic

INTRODUCTION

A lattice point on a two dimensional coordinate plane is one whose both components are integers. A lattice polygon is one whose all the vertices are lattice points. For example, the points p(-1, 1), 0(0, 0) and Q(1, 1) constitute a lattice triangle, whose side PQ = 2 is a natural number but the side OP is not. A lattice triangle is called *Heronian* if its sides and area are all natural numbers. For example, the lattice triangle POQ is not Heronian, but the lattice triangle with vertices at (0, 0), (0, 3) and (4, 0) is Heronian.

Sastry[1] has studied some interesting properties of the lattice triangle inscribed in the lattice parabola $y = x^2$, and raised several related questions for further study.

In this study, we address some of the questions raised by Sastry[1] in connection with the lattice parabola. We also give some interesting properties for the case of the lattice cubic $y = x^3$.

Throughout this study, we denote by N the set of all natural numbers and by Z the set of all intergers, that is,

$$N = \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\},$$
$$Z = \{\ldots\ldots\ldots, -2, -1, 0, 1, 2, \ldots\ldots\ldots\}.$$

LATTICE PARABOLA Y = X^2

In this section, we consider some of the properties of the lattice triangles inscribed in the Lattice Parabola $y = x^2$. The following result is due to Sastry[1]:

Lemma 1: The area of the inscribed (lattice) triangle with vertices at $P(p, p^2)$,

$Q(q, q^2)$ and $R(r, r^2)$ (with $p<q<r$) is $\Delta = (q - p)(r - q)(r - p)$.

From Lemma 1, we see that the area of the lattice triangle with vertices at $P(p, p^2)$, $Q(q, q^2)$ and $R(r, r^2)$ depends only on the differences $q - p$, $r - q$ and $r - p$. Let $q - p = a$, $r - q = b$ (so that $q = p + a$, $r = q + b = p + a + b$) (1)

Then, the area of the triangle PQR is $\Delta = \Delta(a, b) = ab(a + b)$ (2)

Remark 1: From the expression (2) we observe the followings:

- The area is independent of the position of the point P on the lattice parabola and depends only on a and b.
- The area is symmetric with respect ot a and b, that is $\Delta(a, b) = \Delta(b, a)$. Thus, the area of the lattice triangle with vertices at $P(p, p^2)$, $Q(p + a, (p + a)^2)$ and $R(p + a + b, (p + a + b)^2)$ is equal to the area of the triangle whose vertices are P, $Q'(p + b, (p + b)^2)$ and R. Since the triangles PQR and PQ′R are on the common base PR, it follows that QQ′ is parallel to PR.
- $\Delta(a, b)$ is strictly increasing in a (≥ 1) for b (≥ 1) fixed and is strictly increasing in b for a fixed.

Lemma 2: The minimum area of an inscribed triangle is $\Delta_{min} = 1$. Such triangles are those with vertices at (p, p^2), $(p + 1, (p + 1)^2)$, $(p + 2, (p + 2)^2)$ for any $p \in Z$.

Proof: by virtue of part (3) of remark I, $\Delta(a, b) \geq \Delta(1, 1) = 1$ for all $a, b \in N$. Hence, among the inscribed triangles, the minimum ones are those with $a = q - p = 1$, $b = r - q = 1 \Rightarrow q = p + 1$, $r = p + 2$.
We thus get the desired result

Lemma 3: Let $P(p, p^2)$, $Q(q, q^2)$, $R(r, r^2)$ and $S(s, s^2)$ be four points on the lattice parabola. Then, PQ is parallel to R S if and only if

$$p + q = r + s$$

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Further more: if $q > p$, $r > p$, $s > p$ then Q must be the furthest point from p.

Proof: Since slope of PQ = $(q^2 - p^2)/(q - p) = p + q$, it follows that PQ is parallel to RS if and only if $p + q = r + s$.

Next, let $r = p + x$, $s = p + y$, $q = p + z$ for some x, y, $z \in N$.

Then, $z = x + y$, showing that Q is the furthest point from P.

All these complete the proof.

Lemma 4: A parallelogram cannot be embedded in the lattice parabola.

Proof: Let $P(p, p^2)$, $Q(p + a, (p + a)^2)$, $R(p + a + b, (p + a + b)^2)$ and $S(s, s^2)$ be points on the lattice parabola such that QR is parallel to PS. Then, by Lemma 3, $s = p + 2a + b$. But, by Lemma 3, PQ cannot be parallel to RS, since $2p + a \neq 2p + 3a + 2b$.

Hence the lemma.

The following results have been established by Sastry[1].

Lemma 5: An inscribed triangle has a right angle at (p, p^2) if and only if its other vertices are $(-(p + 1), (p + 1)^2)$ and $(P + 1, (p + 1)^2)$.

Lemma 6: The area of the triangle formed by $P(p, p^2)$, $Q(q, q^2)$ an $R(r, r^2)$ is the cube of an integer if and only if p, q and r are in arithmetic progression. We now prove the following result.

Lemma 7: The only (inscribed lattice) triangles, the area of each of which is a square of an integer, are those with vertices at (p, p^2), $(p + 1, (p + 1)^2$ and $(p + 2, (p + 2)^2)$ where $p \in Z$.

Proof : Let

$$\Delta (a, b) = \frac{1}{2} ab(a + b) = \lambda^2, \text{ for some } \lambda \in N.$$

Then, $(ab)^2 \left(\frac{1}{a} + \frac{1}{b}\right) = 2\lambda^2$

Now $\left(\frac{1}{a} + \frac{1}{b}\right) \leq 2$ for all $a, b \in N$,

with the equality sign if and only if $a = b = 1$ and for other values of a and b, $\left(\frac{1}{a} + \frac{1}{b}\right)$ is a fraction. Hence, it follows that we must have $a = b = 1$.

This proves the lemma.

Lemmas 2-7, we note that, the minimum area, triangles are the only triangles whose area can be expressed as a square of an integer.

Now, Δ (n, 1) = n(n + 1), which are the triangular numbers. From lemmas 6, 7 we see that

$\Delta (n, 1) = \lambda^3$ if and only if $n = 1$,

$\Delta (n, 1) = \mu^2$ if and only if $n = 1$,

From (**) and (***), we see that no triangular number greater than 1 is a perfect cube or a perfect squar. As has been noted by Sastry[1], the triangular numbers can be realized as areas of successive triangles issuing from a common vertex. Letting $P_m = (m, m^2)$ be the common vertex, the areas of the triangles in each of the sequences of non-overlapping triangles.

$P_m P_{m+1} P_{m+2}$, $P_m P_{m+2}$ P_{m+3},.................., $P_m P_{m+n} P_{m+n+1}$, ...

and

$P_m P_{m-1} P_{m-2}$, $P_m P_{m-2} P_{m-3}$,..................., $P_m P_{m-n} P_{m-n-1}$, ...

Give the sequence of triangular numbers. We observe further that the triangular numbers can be realized as areas of successive triangles on a common base, such as the sequences: $p_m p_{m+1} p_{m+2}$, $p_m p_{m+2} p_{m+3}$,................, $p_m p_{m+1} p_{m+n+1}$, ...

and $p_m p_{m-1} p_{m-2}$, $p_m p_{m-1} p_{m-3}$,................, $p_m p_{m-1} p_{m-n-1}$, ...

Lemma 8: The only integral solution of the Diophantine equation $X^2 + 1 = y^2$ are $X = 0$, $Y = \pm 1$.

Proof: Writing the Diophantine equation as $(y - x)(y + x) = 1$ and noting that $y - x$ and $y + x$ are both integers, exactly one of the following two cases must hold:

Case 1: $y - x = 1 = y + x \Rightarrow x = 0, y = 1$,

Case 2: $y - x = -1 = y + x \Rightarrow x = 0, y = -1$,

Hence the lemma is proved.

Lemma 9: Let $(p(p, p^2)$ and $Q(q, q^2)$ be two points on the lattice parabola. Then, PQ is an integer if and only if $p + q = 0$.

Proof: Since

$$PQ = \sqrt{(q-p)^2 + (q^2 - p^2)^2} = |q - p|, \sqrt{1 + (q+p)^2}$$

PQ is an integer if and only if $1 + (q + p)^2$ is a perfact square, say, $1 + (q + p)^2 = y^2$.

The assertion of the lemma now follows from Lemma 8.

We are now in a position to prove the following result.

Lemma 10: There is no Heronian triangle inscribed in the lattice parabola.

Proof: If possible, let $P(p, p^2)$, $Q(q, q^2)$ and $R(r, r^2)$ with $(p < q < r)$ be the vertices of a Heronian triangle. Now, since PQ, QR and PR are all integers, by virtue of Lemma 9, $P + q = 0$, $q + r = 0$, $r + p = 0$, Leading to a contradiction and thereby, establishing the lemma.

LATTICE CUBIC $y = x^3$.

This section deals with some properties of the lattice triangles inscribed in the lattice cubic $y = x^3$.

We start with the following lemma.

Lemma 11: The area of the inscribed lattice triangle with vertices at $P(p, q^3)$, $Q(q, q^3)$ and $R(r, r^3)$ (with $p < q < r$) is

$$\Delta = \frac{1}{2}\ (q-p)(r-q)(r-p)\ |p+q+r|$$

Proof: The area of the triangle PQR in terms of the coordinates of its vertices is given by the absolute value of the determinant

$$= \frac{1}{2}\begin{vmatrix} 1 & p & p^3 \\ 1 & q & q^3 \\ 1 & r & r^3 \end{vmatrix},$$

giving the desired expression after some algebraic manipulations. From the above lemma, we see that the points $P(p, p^3)$, $Q(q, q^3)$ and $R(r, r^3)$ on the lattice cubic are collinear if and only if $p + q + r = 0$. It thus follows that any line through two points on the lattice cubic would intersect the lattice cubic at a third point. In particular, the points (-1, -1) (0, 0) and (1, 1) are collinear. We note that the area of the lattice triangle with vertices $P(p, p^3)$, $Q(q, q^3)$ and $R(r, r^3)$ is the same as that of the lattice triangle with vertices $P'(-p, -p^3)$, $Q'(-q, -q^3)$ and $R'(-r, -r^3)$. We further note that the area of the lattice triangle with vertices 0(0, 0), $P(p, p^3)$and $Q(q, q^3)$ is the same as the areas of the lattice triangles with vertices 0, $P'(-p, -p^3)$, Q and 0, P, $Q'(-q, -q^3)$ $P'(-p, -p^3)$, Q and O, P, $Q'(-q, -q^3)$.

Lemma 12: For any $p, q \in z$,

$p^2 + pq + q^2 \geq 0$, where the equality sign holds if and only if $p = q = 0$.

Proof: Writing $p^2 + pq + q^2 = \frac{1}{4}[(2p + q)^2 + 3q^2]$,

We get the desired result.

Lemma 13: The length of the line segment joining any two points on the lattice cubic cannot be an integer and consequently, no Heronian triangle can be formed by joining points on the lattice cubic.

Proof: Let $P(p, p^3)$ $Q(q, q^3)$ be any two points on the lattice cubic.

Then, $PQ = \sqrt{(q-p)^2 + (q^3 - p^3)^2}$

$|q- p| \ = \sqrt{1 + (q^2 + pq + p^2)^2}$

But, by Lemma 12,

$$p^2 + pq + q^2 > 0 \text{ if } p \neq q, \tag{3}$$

So that, by Lemma 8, $(1 + (p^2 + pq + q^2)^2)$ cannot be a perfect square, Hence, PQ cannot be an integer.

The remaining part of the lemma follows from the above result. Since the slope of the line joining the points $P(p, p^3)$ and $Q(q, q^3)$ on the lattice cubic is $p^2 + pq + q^2$, it follows from (3) that the line PQ makes an acute angle with the positive x-axis and further more, the line PQ can never be parallel to the co ordinate axes. Another consequence of (3) is the following.

Lemma 14: No right-angled triangle can be formed by joining three points on the lattice cubic.

Proof: Let $P(p, p^3)$ $Q(q, q^3)$ and $R(r, r^3)$ be three points on the lattice cubic such that the angle PQR is a right angle. Then, we must have $(p^2 + pq + q^2)(q^2 + qr + r^2) = -1$.

But, by (3), each factor on the left hand side of the above relationship is positive. Thus we are led to a contradiction, establishing the lemma.

As a consequence of the above lemma, we see that no square can be embedded in the lattice cubic; in fact, we have the stronger result that no parallelogram can be embedded in that lattice cubic (cf: lemma 4.)

$$\text{Let } q - p = m,\ r - q = n \text{ for some } m, n \in N. \tag{4}$$

Then, by Lemma 11, the area of the lattice triangle with vertices at the points $P(p, p^3)$, $Q(q, q^3)$ and $R(r, r^3)$ on the lattice cubic (with $p < q < r$) is given by

$$\Delta = \Delta(p,\ m,\ n) = ½\ mn(m+n)\ |3p + 2m + n|: \quad m, n \in N, P \in Z \tag{5}$$

From the above expression, we see that the value of the area of the triangle PQR depends on the position of the point P as well as the differences m and n. It may be remarked here that, for p and n fixed, Δ (p, m, n) is not necessarily increasing in m. For example, though

$$\Delta(-2, 2, 3) = 15 > 6 = \Delta(-2, 1, 3),$$

We see that $\Delta(-2, 2, 2) = 0 < 6 = \Delta(-2, 1, 2)$,

Similarly, for p and m fixed, Δ(p, m, n) is not necessarily increasing in n.

Now, since

$\Delta(p, m + 1, n) = ½ (m + 1)n (m + n + 1) [3p + 2m + n + 2]$, after laborious calculations, we would have the following expression
$\Delta(p, m + 1, n)$

$$\begin{cases} \Delta(p,m,n)+\Delta(p+m+n+1), \text{if } 3p+n+2\geq 0 \\ \Delta(p,m,n)-\Delta(p,1,n)+3mn(p+m+n+1), \text{if } -2m\leq 3p+n\leq -2 \\ -\Delta(p,m,n)-\Delta(p,1,n)+3mn(p+m+n+1), \text{if } -2\leq 3p+2m+n\leq 0 \\ \Delta(p,m,n)+\Delta(p,1,n)-3mn(p+m+n+1), \text{if } 3p+2m+n+2\leq 0 \end{cases} \quad (6)$$

The above relation ship gives the expression of $\Delta(p, m + 1, n)$ in terms of $\Delta(p, m. n)$ Similarly, after laborious calculations, we would have the following expression giving $\Delta(p, m, n + 1)$ in terms of $\Delta(p, m, n)$.
$\Delta(p, m, n + 1)$
We now prove the following result.

$$\begin{cases} \Delta(p,m,n)+\Delta(p,m,1)+3mn(2p+2m+n+1)/2, \text{if } 3p+2m+1\geq 0 \\ \Delta(p,m,n)+\Delta(p,m,1)+3mn(2p+2m+n+1)/2, \text{if } -n\leq 3p+2m\leq -1 \\ -\Delta(p,m,n)-\Delta(p,m,1)+3mn(2p+2m+n+1)/2, \text{if } -1\leq 3p+2m+n\leq 0 \\ \Delta(p,m,n)+\Delta(p,m,1)+3mn(2p+2m+n+1)/2, \text{if } 3p+2m+n+1\leq 0 \end{cases} \quad (7)$$

Lemma 15: The area of the triangle formed by any three points $P(p, p^3)$, $Q(q, q^3)$ and $R(r, r^3)$ (with $p < q < r$) on the lattice cubic is a multiple of 3, that is,

$$\Delta(p, m, n) = 3I \text{ where } I \in \{0, 1, 2, \ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\}$$

Proof: We fix p and then prove by double induction on m and n. Since

$$\Delta(p, m, m) = 3m^3|p + m|;\ p \in z,\ m \in N. \quad (8)$$

We see that the lemma is valid when $m = n = 1$. Now, assuming the validity of the result for all m' and n' with $1 \leq m' \leq m$, $1 \leq n' \leq n$, w see, by virtue of (6) and (7), that the result is true for $(m + 1, n)$ and $(m, n +1)$ as well. This completes the induction. We have already mentioned that the points (-1, -1), (0, 0) and (1, 1) on the lattice cubic are collinear (with the area bounded being 0); the next minimum- area triangle (with area 3) is given, by virtue of (8), by $m = n = 1$, $p = 0$, which is the triangle formed by the points (0, 0) (1, 1) and (2, 8) (another triangle of area 3 is obtained by the points (-1, -1), (0, 0) and (-2, -8), However, there is still another triangle of area 3; this correspond to the case when $p = -1$, $m = 1$, $n = 2$ (as may be checked from (5) and the resulting triangle is obtained by the points (-1, - 1), (0, 0) and (2, 8) and hence, the fourth triangle of area 3 is obtained by the points (-2, -8), (0, 0) and (1, 1). However thus, there are four triangles, each having area 3 and it may easily be checked that these are the only triangles. There are six triangles, each of area 6 and these are obtained by the points (1) (1, 1), (2, 8), (3, 27), (2) (-3, -27), (-2, - 8), (- 1, - 1), (3) (-1, -1), (1, 1), (2, 8), (4) (-2, -8) (-1, -1), (1, 1) (5), (-2, -8), (-1, -1), (2, 8), (6) (-2, -8), (1, 1), (2, 8). However, there are only two triangles; namely, those formed y the points (1), (2, 8), (3, 27), (4, 64) and (2) (-4, -64), (-3, -27), (-2, -8), of area 9 each.

Lemma 16: A parallelogram cannot be embedded in a lattice cubic.

Proof: Let $(P(p, p^3)$, $Q(q, q^3)$, $R(r, r^3)$ and $S(s, s^3)$ be points on the lattice cubic with $O < p < q < r < s$, such that QR is parallel to PS. Then,

$$q^2 + qr + r^2 = p^2 + ps + s^2 \quad (9)$$

Now, if PQ is parallel to RS, then

$$p^2 + pq + q^2 = r^2 + rs + s^2 \quad (*)$$

Adding (9) and (*) side-wise, we get, after a bit of algebraic calculations

$$q[(p +q) + (q +r)] = s [(p + s) + (r + s)] \quad (**)$$

But, by our choice
$s > q,\ p + s > p + q,\ r + s > q + r$,
and hence, the relationship (**) is absurd. Consequently, QR cannot be parallel; to PS.

It may be mentioned here that, given any two points $P(p, p^3)$ and $(Q(q, q^3)$ (with $p \neq 0, q \neq 0, - p$) on the lattice cubic, we can always find another line parallel to PQ and intersecting the lattice cubic, namely, the line joining the points P' $(-p, -p^3)$ and Q' $(-q, -q^3)$

Now, let $P(p, p^3)$ $(p \neq 0)$ be any point on the lattice cubic. Then, the line PO would intersect the lattice cubic at the point p' $(-p, -p^3)$. Let $Q(q, q^3)$ and $R(r, r^3)$ (with $q \neq 0, \pm p$, $r \neq 0, \pm p \pm q$) be two points on the lattice cubic such that QR is parallel to POP'. This results in the following Diophantine equation:

$$P^2 = q^2 + qr + r^2, \quad (10)$$

or equivalently,

$$4p^2 = (2q + r)^2 + 3r^2 \quad (11)$$

with the two sets of trivial solutions

$$p = \pm x,\ 2q + r = \pm x,\ r = \pm x,\quad x \in N \quad (11a)$$
$$p = \pm x,\ 2q + r = \pm 2x,\ r = 0,\quad x \in N \quad (11b)$$

We note that, for our purpose, the trivial solutions (11a-b) are not acceptable. Thus the problem of finding the lines parallel to POP' is equivalent to the problem of finding the nontrivial solutions of the Diophantine equation (10). It can be seen that, for p = 1, 2, 3, 4, 5, 6 the trivial solutions are the only solutions of the Dipohantine equation (10); however, for p = 7, the non- trivial solutions are

(1) q = 3, r = 5; (2) q = 3, r = -8; (3) q = 5, r = - 8; (4) q = 8, r = - 5;
(5) q = -3, r = -5; (6) q = -3, r = 8; (7) q = -5, r = 8, (8) q = - 8, r = 5.

From the above solutions, we get two lines, namely, the line through the points (3, 27), (5, 125) and (-8, - 512) and the line through the points (-3, -27), (-5, -125) and (8, 512), each of which is parallel to the line through 0(0,0) and P(7, 343).

CONCLUSIONS

In connection with the lattice parabola, the following problems have been proposed by Sastry[1]:

(1) Characterize two triangles whose areas are in a given ratio,
(2) Determine the parabolic segments whose lattice point content is a square. The problem of finding the lattice point content of the portion of the lattice parabola below one of its chords has been treated by DeTemple[2], using Pick's theorem... In case of the lattice cubic, some problems of interest are given below:
(1) From Lemma 15, we see that
$\Delta(p, m, n) = 3I, \qquad I \in \{0, 1, 2, ...\}$.

For any fixed value of 1, characterize all the triangles whose areas equal 31, that is, determine all p, m and n such that

$$\frac{1}{2} mn(m + n) \,|3p + 2m + n|\, = 3I$$

We have seen that, for I = 1, 2, 3, there are, respectively four, six and two triangles. Since $\Delta(p, m, n) = 3m^3|p + m| = 3I$

We see that two triangles, each of area 3 I, correspond to
m = 1,p = 1 – m and m = 1, p = - (1 +m).

(2) Characterize all the triangles whose area is a square, that is, determine all p, m and n such that $\Delta(p, m, n) = \mu^2$.

By Lemma 15, 3 must divide μ, that is, μ must be of the form 3I, I ∈ N.

(3) Characterize all the triangles whose area is a cube, that, determine all p, m and n such that

$\Delta(p, m, n) = \lambda^3$. Here also, by Lemma 15, λ must be of the form 3I, I ∈ N.

(4) Determine all the lines parallel to the line through 0(0, 0) and $P(p, p^3)$, p ∈ N. We have shown that, for p = 1, 2, 3, 4, 5, 6, there are no other lines, while for p = 7, there are exactly two lines, each parallel to OP. Clearly, for p = 7 I, with I ∈ N, we would have two lines parallel to OP and the problem is to find other values of p with such a property. As has already been stated, the problem of finding lines parallel to OP is equivalent to the problem of solving the Diophantine equation (10).
(5) Corresponding to any line through the points (p, p^3), (q, q^3) and $(-(p+q), -(p+q)^3)$ with $p \neq 0$, $q \neq 0$, there exists a line parallel to the original line, namely, the line through the points $(-p, -p^3)$ and $(-q, -q^3)$. The question is: Is there any third line parallel to these two lines?

REFERENCES

1. Sastry, K.R.S., 1991. The College Math. J., 22: 301-306.
2. De Temple, D. and J.M. Robertson, 1977. Math. Magazine, 50: 152-158.