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An Improved Real-Time Adaptive Constrained Quaternion Extended Kalman Filter

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Abstract: In this study, a new improved real time Adaptive Constrained Quaternion Extended Kalman Filter (ACQEKF) algorithm is proposed. It is employed to estimate the quaternion and bias states of a constrained nonlinear system perturbed by noise using noisy measurements. The values of the process and measurement noise covariances Q and R, respectively are unknown or partially known, their biased initializations result in the degradation or divergence of the quaternion Extended Kalman Filter (EKF) performance. This study proposes a new method to improve the EKF performance against the covariances uncertainty. Unlike, the previous methods, this method adopts the idea of the recursive estimation of the EKF to propose two tunable recursive updating rules for Q and R, respectively designed based on the filter innovations. As for the quaternion constraint, it is projected onto the EKF gain derivation. The proposed ACQEKF proved itself to have a dramatic improved performance over the conventional EKF, the estimates are more accurate have less noise and more stable.

INTRODUCTION

THE state of art of estimation based on the Extended Kalman Filter (EKF) is one of the most famous estimation tools for nonlinear systems. It incorporates the observer theory and the Bayesian approach. The EKF uses noisy measurements to estimate the states of a dynamic system perturbed by noise[1].

The EKF is used with the quaternion representation in many applications to estimate the orientation of a rigid body using the readings of a gyroscope. One problem is that the quaternion is constrained to unity norm and this is not preserved by the EKF[2]. To overcome this problem, numerical techniques are applied on the post-estimated quaternion to maintain its unity norm. Here, a systematic method of including the unity constraint into the filter derivation[3] is proposed, this filter is called constrained quaternion extended Kalman filter (CQEKF). For this CQEKF, the filter gain is derived based on minimizing the state covariance subjected to the unity constraint.

Another problem is related to the noise models. The structure of the EKF is composed of the plant dynamic nonlinear model which describes the system behavior over time and the stochastic models which describe both the process and observation noises properties[3]. The EKF uses the noise statistics to influence the EKF gain that is applied on the filter innovation error which affects the estimation performance. Thus, the EKF qualification depends on the knowledge of the stochastic models parameters. Thus the EKF qualification depends on the knowledge of the stochastic models parameters. Therefore, The uncertain model parameters will affect the EKF performance adversely. Further, this performance degrades or may even diverge[4, 5]. Therefore, improving the EKF such that it can adapt itself to the uncertainty in the noise statistical parameters and reduce their effects is of significant importance. The most used adaptive EKF schemes in the literature can be divided as: Innovation-based Adaptive Estimation (IAE), Multiple Model Adaptive Estimation (MMAE) and a scaled state covariance method. A summary of the first two methods can be found by Mohamed and Schwarz[6].

The IAE estimation method assumes that the innovation sequence of the EKF is white noise. Based on this assumption it estimates the process noise covariance matrix R and/or the measurement covariance matrix Q. One technique in this method is the covariance matching technique, it assumes the availability of large window of data to compute the sample covariance which is employed to estimate Q and/or R[7-9]. Another technique is correlation one, it uses the sample auto-correlations between the innovations to estimate the covariances[10-12]. Both of the above techniques require large window of data which makes them impractical. The use of the maximum likelihood techniques are used to estimate the covariances by maximizing the likelihood function of the innovations[13]. However, they can be implemented off-line their computations are heavy a modified one with using the Expectation-Maximization algorithm (EM) is reported by Bavdekar *et al*.[14].

The MMAE method is model based: It assumes the availability of the correct model among a bank of different available models. Using the measurement sequence, the Bayesian probability is computed for each model as it is the true model. Later, the output of the highest probability model or the output of the weighting sum of all models is considered. However, having the correct model assumption makes it unsuitable for the uncertain dynamic systems[15]. Scaling the error state covariance matrix by a factor is reported to improve the filter stability and convergence performance[16, 9]. The scaling can be empirical.

The main aim here is designing an adaptive CQEKF (ACQEKF) that avoids the aforementioned problems of using moving window, excessive computations, exact models and scaling the state error covariance matrix. Furthermore, the ACQEKF core concept is to be able to adapt itself to the biased initial covariances smoothly while running, to increase the estimation accuracy and to stabilize the filter if the used noise covariance values cause filter divergence.

In this study, a new ACQEKF is proposed to achieve the above advantages. The unity constraint is included in the derivation of the filter. The ACQEKF adopts the idea of the recursive estimation of the EKF to add two tunable recursive unbiased updating rules for the noise covariances Q and R. Each rule consists of a scaled value of the previous covariance matrix and a correction covariance error term. This error is calculated at each sample time by using the most recent measurements and innovations along with the available information about the state covariance error. The updating rules tunes the matrices Q and R to get the best performance.

MATERIALS AND METHODS

Consider the discrete-time nonlinear state space model:

$$x_k = f\left(x_{k-1}, u_{k-1}\right) + \Gamma_{k-1}\upsilon_{k-1} \quad (1)$$

$$y_k = h\left(x_k\right) + v_k \quad (2)$$

where, $x \in n$ is the state vector, $y \in d$ is the measurement vector and k is the time index. u is the input and assumed to be piecewise constant over the sampling time intervalT and the process and measurements have the same sampling time. $v_{k-1} \in m$ and $v_k \in d$ are the Gaussian process and measurement noises, respectively, they are assumed to be independent and mutually uncorrelated with constant covariances Q and R, respectively. $\Gamma \in n\times m$ maps the noise to the states space. The state estimation is carried out under the following assumptions. Then, for the given system in (1), the conventional EKF algorithm is composed of the prediction step:

$$\hat{x}_k^- = f\left(\hat{x}_{k-1}, u_{k-1}\right) \quad (3)$$

$$P_k^- = A_{k-1}P_{k-1}A_{k-1}^T + \Gamma_{k-1}Q_{k-1}\Gamma_{k-1}^T \quad (4)$$

and the measurement update step:

$$S_k = H_k P_k^- H_k^T + R_k \quad (5)$$

$$K_k = P_k^- H_k^T S_k^{-1} \quad (6)$$

$$e_k = z_k - h\left(\hat{x}_k^-\right) \quad (7)$$

$$\hat{x}_k = \hat{x}_k^- + K_k e_k \tag{8}$$

$$P_k = (I - K_k H_k) P_k^- \tag{9}$$

where:

$$A_{k-1} = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1}, u_{k-1}}, H_k = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_k^-} \tag{10}$$

Equation 3-9 the following notation is employed. $(.)^-$ and (.) stand for the prior and posterior estimates, respectively. K is the Kalman gain. I is the identity matrix and P is the estimation error covariance matrix, x is the estimated state and z is the measurement vector with the same dimension as y.

Remarks:

- For the quaternion estimation $H_k = [I_4 \quad 0_{4\times3}]$
- The Kalman gain in Eq. 6 can be rewritten as:

$$K_k = P_k H^T R_k^{-1} \tag{11}$$

Unit quaternion constraint projection: The quaternion $q = [q_0\ q_1\ q_2\ q_3]^T \in 4$ with the constraint of unit $l^2{}_2$ norm ($\|q\|^2{}_2 = 1$) has an important role in representing rotations and orientation of a rigid body with respect to a reference frame[17]. It is generally used with the readings of the gyro-rate due to the direct relation between the quaternion time derivative q and the gyroscope angular velocity $\Omega\in 3$ as:

$$\dot{q} = q \otimes \Omega \tag{12}$$

where $\otimes$ is the quaternion multiplication. Equation 12 is nonlinear and the gyroscope angular velocity has bias $b\in 3$ band contaminated white zero mean noise $v\in 3$ as stated in:

$$\Omega = \omega + b + v \tag{13}$$

The bias is modeled as an integrated white noise $v_b \in 3$ as:

$$b_k = b_{k-1} + v_{b,\,k-1} \tag{14}$$

Due to the bias and noise, the quaternion is estimated by employing the EKF. Defining the state vector as $x_k = [q^T{}_k\ b^T{}_k]^T$ and taking the discrete form of Eq. 12 along with Eq. 13 and 14, then the model (1) is obtained with:

$$f(x_{k-1}, u_{k-1}) \equiv \begin{bmatrix} \left(I_4 + \frac{1}{2} T U(\Omega_{k-1} - b_{k-1})\right) q_{k-1} \\ b_{k-1} \end{bmatrix} \tag{15}$$

$$\upsilon_{k-1} \equiv \begin{bmatrix} \mathbf{v}_{k-1} \\ \mathbf{v}_{b,k-1} \end{bmatrix} \tag{16}$$

where:

$$U(\chi) = \begin{bmatrix} 0 & -\chi_x & -\chi_y & -\chi_z \\ \chi_x & 0 & \chi_z & -\chi_y \\ \chi_y & -\chi_z & 0 & \chi_x \\ \chi_z & \chi_y & -\chi_x & 0 \end{bmatrix} \tag{17}$$

$$\Gamma_{k-1} = \begin{bmatrix} -\frac{1}{2} T \bar{U}(q_{k-1}) & 0_{4\times3} \\ 0_{3\times3} & I_3 \end{bmatrix} \tag{18}$$

and:

$$\bar{U}(\chi) = \begin{bmatrix} -\chi_1 & -\chi_2 & -\chi_3 \\ \chi_0 & -\chi_3 & \chi_2 \\ \chi_3 & \chi_0 & -\chi_1 \\ -\chi_2 & \chi_1 & \chi_0 \end{bmatrix} \tag{19}$$

This model is used in the EKF algorithm. The output of the EKF is q. One main problem in the quaternion estimation is to maintain its unity norm which is not preserved by the EKF. Usually, post estimation numerical correction is used as $\tilde{q}_N = \tilde{q} / |\tilde{q}|_2^2$ where $\tilde{q} = \hat{x}$ in Eq. 8 and $\tilde{q}_N$ is the normalized value of $\tilde{q}$ or as presented by the author $\tilde{q}_N = \tilde{q}(1 - \|\tilde{q}\|_2^2)\tilde{q}$.

Though it is a low cost method of unity preserving; it is done out of the EKF derivation. Here, the unity norm constraint is projected on the Kalman gain derivation. The unity norm constraint is expressed as:

$$f(q) = \|q\|_2^2 = \sum_{i=0}^{3} q_i^2 = 1 \tag{20}$$

however, since, the true value of the quaternion vector is unknown, the predicted quaternion is used in Eq. 20 and linearized using Taylor series as:

$$f(\hat{q}_k^-) = f(\hat{q}_k^-) + \dot{f}(\hat{q}_k^-)(\hat{q}_k - \hat{q}_k^-) + \text{hot} = 1 \tag{21}$$

In Eq. 21, the term is the higher order terms which are neglected, then Eq. 21 has only q as unknown and can be expressed as:

$$G\hat{q} = d \tag{22}$$

where:

$$G = \dot{f}(\hat{q}_k^-) = \frac{\partial f(\hat{q}_k^-)}{\partial(\hat{q}_k^-)} = 2\left[\hat{q}_0^- \quad \hat{q}_{1,}^- \quad \hat{q}_2^- \quad \hat{q}_3^-\right] \tag{23}$$

and:

$$d = 1 + \sum_{i=0}^{3} \left(\hat{q}_k^-\right)_i^2 \quad (24)$$

The unconstraint Kalman filter gain is given K_k by Simon[3]:

$$K_k = \arg\min_{K_k} Tr\begin{pmatrix} (I - K_k H_k) P_k^- (I - K_k H_k)^T \\ + K_k R_k K_k^T \end{pmatrix} \quad (25)$$

where T_r stands for the trace. The result of Eq. 25 is the gain solution and update as in (6). In the same way, minimizing (25) subjected to the constraints $Gq_k = d$ yields to the constrained Kalman gain K_k[3]:

$$\widehat{K}_k = K_k - \begin{bmatrix} \rho_k \\ 0_{3\times4} \end{bmatrix} \quad (26)$$

where K_k is given in Eq. 6 and ρ is:

$$\rho_k = G^T (GG^T)^{-1} (G\hat{q}_k - d)(e_k S_k^{-1} e_k^T)^{-1} e_k^T S_k^{-1} \quad (27)$$

As a final estimation, the CQEKF follows Eq. 4-10 with 28 and replaces K_k by K_k in Eq. 9:

$$\hat{x}_k^+ = \hat{x}_k - \begin{bmatrix} \rho_k \\ 0_{3\times4} \end{bmatrix} e_k \quad (28)$$

Adaptive CQEKF: The values of Q and R have an important effect on the EKF performance. Too small or large values of these covariances with respect to the true value results in estimation degradation[18]. Here, two tunable recursive updating rules R1 and R2 are developed to update both Q and R to form the adaptive CQEKF (ACQEKF).

By definition, the ACQEKF is designed to be able to adapt itself to the noise covariance uncertainty in order to achieve better performance. So, increasing this uncertainty by assuming that the expectation of $\Gamma_{k-1} Q_{k-1} \Gamma_{k-1}^T$ can be approximated by $\tilde{Q}_{k-1}$ is still fruitful. This is justified using the fact that, for high sampling frequency, the components of the unit quaternion are always constrained; making drastic changes in the noise effect impossible. Thus, the new model is:

$$\begin{aligned} x_k &= f(x_{k-1}, u_{k-1}) + w_{k-1} \\ y_k &= Hx_k + v_k \end{aligned} \quad (29)$$

where wN (0, Q). For the given system (Eq. 29) and for a given initial value matrices $R_0>0$ and $Q_0>0$ and selected positive constants N_R and N_Q, there are noise covariance errors ΔQ and ΔR such that the CQEKF performance is improved by updating the observation and the process covariance matrices recursively as in Eq. 35 and 46, respectively. The new ACQEKF algorithm is:

$$\hat{x}_k^- = f(\hat{x}_{k-1}^+, u_{k-1}) \quad (30)$$

$$P_k^- = A_{k-1} P_{k-1} A_{k-1}^T + \tilde{Q}_{k-1} \quad (31)$$

$$e_k = z_k - H\hat{x}_k^- \quad (32)$$

$$\bar{e}_k = \frac{N_R - 1}{N_R} \bar{e}_{k-1} + \frac{1}{N_R} e_k \quad (33)$$

$$\Delta R_k = \frac{1}{N_R - 1} (e_k - \bar{e}_k)(e_k - \bar{e}_k)^T - \frac{1}{N_R} H_k P_k^- H_k^T \quad (34)$$

$$R_k = \left| diag\left(\frac{N_R - 1}{N_R} R_{k-1} + \Delta R_k \right) \right| \quad (35)$$

$$S_k = HP_k^- H^T + R_k \quad (36)$$

$$K_k = P_k^- H_k^T S_k^{-1} \quad (37)$$

$$\rho_k = G^T (GG^T)^{-1} (G\hat{q}_k - b)(e_k S_k^{-1} e_k^T)^{-1} e_k^T S_k^{-1} \quad (38)$$

$$\widehat{K}_k = K_k - \begin{bmatrix} \rho_k \\ 0_{3\times4} \end{bmatrix} \quad (39)$$

$$\hat{x}_k = \hat{x}_k^- + K_k e_k \quad (40)$$

$$\hat{x}_k^+ = \hat{x}_k - \begin{bmatrix} \rho_k \\ 0_{3\times4} \end{bmatrix} e_k \quad (41)$$

$$P_k = (I - \widehat{K}_k H) P_k^- \quad (42)$$

$$\hat{\omega}_k = \hat{x}_k^+ - \hat{x}_k^-, \quad (43)$$

$$\bar{\omega}_k = \frac{N_Q - 1}{N_Q} \bar{\omega}_{k-1} + \frac{1}{N_Q} \hat{\omega}_k \quad (44)$$

$$\Delta\tilde{Q}_k = \begin{pmatrix} \frac{1}{N_Q} (P_k - A_{k-1} P_{k-1} A_{k-1}^T) + \\ \frac{1}{N_Q - 1} (\hat{\omega}_k - \bar{\omega}_k)(\hat{\omega}_k - \bar{\omega}_k)^T \end{pmatrix} \quad (45)$$

$$\tilde{Q}_k = \left|\text{diag}\left(\frac{N_Q - 1}{N_Q}\tilde{Q}_{k-1} + \Delta\tilde{Q}_k\right)\right| \quad (46)$$

$$A_{k-1} = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1}^{+}, u_{k-1}}$$

Remarks:

- The ACQEKF converges to the conventional CQEKF if the selected values of N_R and $N_Q \to \infty$
- The update rules keep the noise covariance matrices Q and R positive definite for all k

Process covariance matrix update rule (R1) proof: The true values of the system states x_k and x_{k-1} are unknown which makes it impossible to determine w_{k-1} in Eq. 29. The best known states values are x^+ and the estimated but not updated state x^-. Using these estimates, w is approximated by $w \approx \acute{\omega}_k = x^+_k - x^-_k$[7].

To simplify the proof, assume that a buffer of N_Q samples is available. These samples are ordered from the oldest to the newest. The oldest is located in the buffer index i = 1 and the newest is located in i = N_Q. This can be stated in terms of time instants as follows: the oldest available data is stored in location k-N_Q and the newest is in the time index k. Note that for the available samples the noise is assumed independent, then the sample covariance C_s is:

$$C_s = \frac{1}{N_Q - 1}\sum_{i=k-N_Q}^{k}(\hat{\omega}_i - \overline{\omega})(\hat{\omega}_i - \overline{\omega})^T \quad (47)$$

where:

$$\overline{\omega} = \frac{1}{N_Q}\sum_{i=k-N_Q}^{k}\hat{\omega}_i \quad (48)$$

By using the expectation of the state error covariances P^- and P from the CQEKF and the sample covariance (Eq. 47), one can write as Myers and Tapley[7]:

$$\tilde{Q}_k = C_s + \left(\frac{1}{N_Q}\sum_{i=k-N_Q}^{k} P_i - A_{i-1}P_{i-1}A_{i-1}^T\right) \quad (49)$$

Equation 49 can be divided into two terms, the first term contains all the samples from i = k-N_Q up to i = k-1 and the other term contains only the most recent sample arrived at the time instant k. Accordingly, (Eq. 49) is rewritten as (after some mathematical manipulation):

$$\tilde{Q}_k = \frac{N_Q - 1}{N_Q}X + \Delta\tilde{Q}_k \quad (50)$$

where:

$$X = \frac{N_Q}{(N_Q - 1)^2}\sum_{i=k-N_Q}^{k-1}(\hat{\omega}_i - \overline{\omega})(\hat{\omega}_i - \overline{\omega})^T + \left(\frac{1}{N_Q - 1}\sum_{i=k-N_Q}^{k-1} P_i - A_{i-1}P_{i-1}A_{i-1}^T\right) \quad (51)$$

and:

$$\Delta\tilde{Q}_k = \frac{1}{N_Q - 1}(\hat{\omega}_k - \overline{\omega}_k)(\hat{\omega}_k - \overline{\omega}_k)^T + \frac{1}{N_Q}\left(P_k - A_{k-1}P_{k-1}A_{k-1}^T\right) \quad (52)$$

The term X in Eq. 51 depends on the samples up to the sample k-1. Taking the sample covariance of the samples up to the sample:

$$k - 1\ C_{s_{k-1}} = \frac{1}{N_Q - 2}\sum_{i=k-N_Q}^{k-1}(\hat{\omega}_i - \overline{\omega})(\hat{\omega}_i - \overline{\omega})^T$$

and compare it with the first term in X, if the value of $N_Q/(N_Q-1)^2$ can be approximated by $1/(N_Q-2)$, then this term is the noise covariance of the samples up to i = k-1. Let's define the error δ as:

$$\delta = \frac{N_Q}{(N_Q - 1)^2} - \frac{1}{N_Q - 2} \quad (53)$$

this error converges to zero as the number of samples increase $\left(\lim_{N_Q \to \infty} \delta \to 0\right)$. Then, for large value of N_Q, (Eq. 51) can be approximated by:

$$X \approx \frac{1}{N_Q - 2}\sum_{i=k-N_Q}^{k-1}(\hat{\omega}_i - \overline{\omega})(\hat{\omega}_i - \overline{\omega})^T + \left(\frac{1}{N_Q - 1}\sum_{i=k-N_Q}^{k-1} P_i - A_{i-1}P_{i-1}A_{i-1}^T\right) \quad (54)$$

this is the process covariance for the N_Q-1 samples ordered from i = k-N_Q to i = k-1, i.e., $\tilde{Q}_{k-1}$. The same method is used to compute as:

$$\overline{\omega}_k = \frac{1}{N_Q}\sum_{i=k-N_Q}^{k}\hat{\omega}_i = \frac{1}{N}\sum_{i=k-N_Q}^{k-1}\hat{\omega}_i + \frac{1}{N_Q}\hat{\omega}_k \quad (55)$$

this yields:

$$\overline{\omega}_k = \frac{N_Q - 1}{N_Q}\overline{\omega}_{k-1} + \frac{1}{N_Q}\hat{\omega}_k \quad (56)$$

The positive definiteness in the updated values is not guaranteed; this explains the reason of taking the absolute value of the diagonal in Eq. 35 and 46.

Observation covariance matrix update rule (R2) proof: The observation covariance can't be estimated using (Eq. 2) because the true values of the states are unknown. Therefore, the estimation innovation in Eq. 7 and its covariance S given in Eq. 5 are used for this purpose. The proof follows the same procedure as in the previous section. For N_R available samples, the sample covariance of e_k and the expectation of the covariance S can be use to obtain the observation noise covariance[7]:

$$R_k = \left\{ \begin{array}{l} \frac{1}{N_R - 1} \sum_{i=k-N_R}^{k} (e_i - \bar{e})(e_i - \bar{e})^T - \\ \frac{1}{N_R} \sum_{i=k-N_R}^{k} \left(HP^- H^T \right)_i \end{array} \right\} \quad (57)$$

where:

$$\bar{e} = \frac{1}{N_R} \sum_{i=k-N_R}^{k} e_i \quad (58)$$

When the same approximation in R1 is used here too, and the result is a proof R2 and it ends up with Eq. 35.

Stability of ACQEKF: The exponential behavior of the observer:

$$\hat{x}_k^- = f\left(\hat{x}_{k-1}^+, u_{k-1}\right) \quad (59)$$

$$\hat{x}_k^+ = \hat{x}_k^- + \widehat{K}_k \left(Hx_k - H\hat{x}_k^-\right) \quad (60)$$

is determined based on the exponential convergence of the dynamic error $\varepsilon_k = x_k - x_k^+$. First Taylor expansion for Eq. 29 and 59 with nonlinear functions ∂_1 and ∂_2 is written as:

$$\hat{x}_k^- = f\left(\hat{x}_{k-1}^+, u_{k-1}\right) = A_{k-1}\hat{x}_{k-1}^+ + \vartheta_1\left(\hat{x}_{k-1}^+, u_{k-1}\right) \quad (61)$$

$$x_k = f\left(x_{k-1}, u_{k-1}\right) = A_{k-1}x_{k-1} + \vartheta_2\left(x_{k-1}, u_{k-1}\right) \quad (62)$$

The dynamic error ε, after mathematical manipulation, can be expressed as:

$$\varepsilon_k = \left(A_{k-1} - \widehat{K}_k H A_{k-1}\right)\varepsilon_{k-1} + \varphi_k \quad (63)$$

where:

$$\varphi_k = \left(I - \widehat{K}_k H\right)\left(\vartheta_2\left(x_{k-1}, u_{k-1}\right) - \vartheta_1\left(\hat{x}_{k-1}^+, u_{k-1}\right)\right) \quad (64)$$

The exponential stability is proven here based on Lyapunov function theory and follows the approach as in Babacan *et al*.[19]. The following definitions and lemmas are employed for the sake of completeness and proof.

Definition 1; Khalil[20]: The origin of the difference (Eq. 63) is exponentially stable equilibrium point if there is continuous differentiable positive definite function $V(\varepsilon_k)$ such that:

$$\begin{array}{l} c_1 \left\| \varepsilon_k \right\|^2 \le V\left(\varepsilon_k\right) \le c_2 \left\| \varepsilon_k \right\|^2, \\ \Delta V\left(\varepsilon_k\right) \le -c_3 \left\| \varepsilon_k \right\|^2 \end{array} \quad (65)$$

for positive constants c_1-c_3 with ΔV as the rate of change of V and defined by:

$$\Delta V = V\left(\varepsilon_k\right) - V\left(\varepsilon_{k-1}\right) \quad (66)$$

For sake of completeness, the exponential stability for discrete time systems is defined by the inequality $\left\|\varepsilon_k\right\| \le \beta \left\|\varepsilon_0\right\| \Upsilon^k$ for all $k \ge 0$ with $\beta > 0$ and $0 < Y < 1$[20].

Definition 2: The observer in Eq. 60 is an exponential observer if Eq. 63 has an exponentially stable equilibrium at 0.

Definition 3: If A_{k-1} is an invertible matrix for all k and for the positive definite matrices P_k^- and P_k, then:

$$\begin{array}{l} P_k^{-1} \le \left(I - K_k H\right)^{-T} A_{k-1}^{-T} \times \\ \quad \left(P_{k-1}^{-1} - P_{k-1}^{-1}\left(P_{k-1}^{-1} + A_{k-1}^T Q_{k-1}^{-1} A_{k-1}\right)^{-1} P_{k-1}^{-1}\right) \times \\ \quad A_{k-1}^{-1}\left(I - K_k H\right)^{-1} \end{array} \quad (67)$$

Proof: Hashlamon and Erbatur.

Lemma 1: Consider the real and bounded system states $x_k \equiv [q^T{}_K]^T$, the matrix K and the nonlinear functions ∂_1 and ∂_2 such that the following assumptions hold:

- The matrices $\|A\| \le a$, $\|K\| \le k$ and $\|H\| \le \hbar$ are bounded by the positive real numbers $a\hbar$, $k > 0$ for every time instant k
- There are positive real numbers k_∂, σ_∂, $\sigma > 0$ such that:

$$\left\|\vartheta_2\left(x\;, u\;\right) - \vartheta_1\left(\hat{x}^+, u\;\right)\right\| \le \kappa_\vartheta \left\|x\; - \hat{x}^+\right\|^2 \quad (68)$$

holds for $\left\|x\; - \hat{x}^+\right\| \le \sigma_\vartheta < \frac{1}{2}\sigma.$

The stability theorem: The given system in Eq. 29 with the proposed ACQEKF, is exponentially stable if the following assumptions hold. There are positive real numbers p, p>0 such that they bound the following matrices for every time instant k:

$$\underline{p}I \le P_{k-1} \le \overline{p}I \quad (69)$$

$$\underline{p}I \le P_k \le \overline{p}I \quad (70)$$

The matrices Q and R are positive definite due to the updating rules for all k with minimum eigenvalues q>0 and r>0, respectively. There are positive real numbers k_∂, σ_∂, σh>0 such that:

$$\left\|\vartheta_2(x, u) - \vartheta_1(\hat{x}^+, u)\right\| \le \kappa_\vartheta \left\|x - \hat{x}^+\right\|^2 \quad (71)$$

holds for $\left\|x - \hat{x}^+\right\| \le \sigma_\vartheta < \frac{1}{2}\sigma.$ The matrix A_{k-1} is nonsingular for all k.

Proof: Consider the positive definite Lyapunov function:

$$V(\varepsilon_{k-1}) = \varepsilon_{k-1}^T P_{k-1}^{-1} \varepsilon_{k-1} \quad (72)$$

with V(0) = 0. Equation 69 and 72 imply that:

$$\frac{1}{\overline{p}}\left\|\varepsilon_{k-1}\right\|^2 \le V(\varepsilon_{k-1}) \le \frac{1}{\underline{p}}\left\|\varepsilon_{k-1}\right\|^2 \quad (73)$$

Then for $V(\varepsilon_k)$, we obtain:

$$V(\varepsilon_k) = \varepsilon_k^T P_k^{-1} \varepsilon_k \quad (74)$$

Substituting (Eq. 63) into (Eq. 74), we get:

$$V(\varepsilon_k) = \begin{pmatrix} \left((A_{k-1} - \widehat{K}_k H A_{k-1})\varepsilon_{k-1} + \varphi_k\right)^T \times \\ P_k^{-1}\left((A_{k-1} - \widehat{K}_k H A_{k-1})\varepsilon_{k-1} + \varphi_k\right) \end{pmatrix} \quad (75)$$

The assumptions (Eq. 69), (Eq. 70) and A4 imply that P^-_k, P_k and A_{k-1} are respectively non-singular. Thus, the requirements of definition 3 are fulfilled, then by using (Eq. 67) together with (Eq. 75) yield:

$$\begin{aligned} V(\varepsilon_k) \le (\varepsilon_{k-1})^T \left(\left(P_{k-1}^{-1} - P_{k-1}^{-1}\left(P_{k-1}^{-1} + A^T Q_{k-1}^{-1} A\right)^{-1} P_{k-1}^{-1}\right)\right)\varepsilon_{k-1} \\ + \left((A_{k-1} - \widehat{K}_k H A_{k-1})\varepsilon_{k-1}\right)^T P_k^{-1}\varphi_k \\ + (\varphi_k)^T P_k^{-1}(A_{k-1} - \widehat{K}_k H A_{k-1})\varepsilon_{k-1} + (\varphi_k)^T P_k^{-1}\varphi_k \end{aligned} \quad (76)$$

Applying the bounds in A1 and using the eigenvalues in A2 along with (Eq. 11) yields:

$$\|K_k\| \le \|P_k\|\left\|H^T\right\|\left\|R_k^{-1}\right\| \le \kappa \quad (77)$$

where:

$$\kappa = \frac{\overline{p}\overline{h}}{\underline{r}}$$

Now applying Lemma 1 along with A1, A3 and (Eq. 77) on (Eq. 76), we get:

$$\Delta V \le -\overline{\kappa}\left(\sigma - \|\varepsilon_{k-1}\|\right)\|\varepsilon_{k-1}\|^2 \quad (78)$$

where:

$$\overline{\kappa} = \frac{\kappa_\vartheta}{\underline{p}}\left(2a(1+\kappa\overline{h}) + \kappa_\vartheta\sigma_\vartheta\right)$$

and:

$$\sigma = \frac{1}{\overline{p}^2\overline{\kappa}\left(\frac{1}{\underline{p}} + \frac{a^2}{\underline{q}}\right)}$$

Then, with:

$$\|\varepsilon_{k-1}\| \le \sigma_\vartheta \le \frac{1}{2}\sigma$$

it follows that:

$$\Delta V \le -\frac{1}{2\overline{p}^2\left(\frac{1}{\underline{p}} + \frac{a^2}{\underline{q}}\right)}\|\varepsilon_{k-1}\|^2 \quad (79)$$

holds for $\|\varepsilon_{k-1}\| \le \sigma_\partial$ which satisfies (Eq. 65) and thus the origin of (Eq. 63) is exponentially stable. In terms of states and performance, using (Eq. 78) and (Eq. 73), we can write:

$$\|\varepsilon_k\| \le \sqrt{\frac{\overline{p}}{\underline{p}}}\left(\sqrt{1 - \frac{\underline{p}}{2\overline{p}^2\left(\frac{1}{\underline{p}} + \frac{a^2}{\underline{q}}\right)}}\right)^k \|\varepsilon_0\|, k \ge 0 \quad (80)$$

Recalling definition 1, we have

$$\beta = \sqrt{\frac{\overline{p}}{\underline{p}}} > 0$$

and:

$$\Upsilon = \sqrt{1 - \frac{\underline{p}}{2\overline{p}^2\left(\frac{1}{\underline{p}} + \frac{a^2}{\underline{q}}\right)}}, 0 < \Upsilon < 1$$

RESULTS AND DISCUSSION

In this part, both the EKF and the ACQEKF are tested and compared. The bias and process noises with the covariance Q_{true} are added to the angular velocity measured from the gyro-rate sensor, this forms Ω which is the input u for (Eq. 1) and (Eq. 29). The measurement noise with covariance R_{true} is added to the measured quaternion to form the measurement vector z in Eq. 7. The Gaussian noise is generated by the MATLAB simulink Gaussian noise generator. Both of the filters have the same initial values x_0 and P_0. The model Eq. 15-8 are used in the conventional EKF. The ACQEKF uses the model (Eq. 29) and follows the algorithm (Eq. 30-46). The true bias values are chosen to be time t dependent as in Eq. 81:

$$b = \begin{cases} [0.5 \quad 0.1 \quad 0]^T & t \leq 20 \\ [1 \quad 2 \quad 1]^T & 20 < t \leq 50 \\ [0 \quad 0 \quad 2]^T & \text{else} \end{cases} \tag{81}$$

The simulation parameters, initializations and the corresponding numbers of N_Q and N_R are listed in Table 1. Note that the values of N_Q and N_R are different from each other since they don't have to be the same in practice. And since, the true observation noise is much smaller than the process noise, the value of is selected to be much smaller than N_Q as tabulated in Table 1. The following notations are employed: Q_{small} and Q_{big} indicate that the considered process covariance noise in the EKF is either smaller or larger than the true process covariance noise Q_{True}, respectively. The same definition goes for R_{small} and R_{big}. The notations Q_{initid} and $R_{initial}$ refer to the initial values of the covariances used in the ACQEKF. I_n is an n×n identity matrix. The ACQEKF displays its ability to preserve the unit quaternion constraint as depicted in Fig. 1 for the long run of 500 sec. This also shows that though sudden changes in the bias take place, the norm still unity.

Noise covariances have several scenarios, among them is that the used noise covariances are smaller or larger than the true covariances. For example, the values Q_{small}, Q_{big} and R_{big} in Table 1 are used with the EKF, an example of the estimation error is shown in Fig. 2 for the first bias element b_1 and the other states have the same behavior. Figure 2a shows the estimation error for b_1 using the EKF with Q_{small} and R_{small} replacing the small noise covariances values by Q_{big} and R_{big} results in estimation error as displayed in Fig. 2b. As clear that both small and big covariances have noisy behavior. This noise is smoothed when the ACQEKF is used with the same noise covariances as the EKF as shown in Fig. 2c. Moreover, whether the initial covariances are small or big they converge to the same result unlike the EKF.

Another scenario is that when one of the covariances is small while the other is big. For this case, the estimation error for b_2 using the EKF with Q_{big} and R_{small} is noisy as depicted in Fig. 3a, however, Q_{small} and R_{big} show a smooth estimation using the EKF as in Fig. 3b.

Table 1: Initialization and simulation parameters

Parameters	Values
T	0.01 sec
R_{true}	10^{-6} I_4
Q_{true}	10^{-2} I_6
P_0	10 I_7
q_0	$[0.5\ 0.5\ 0.5\ 0.5]^T$
b_0	$[0\ 0\ 0]^T$
N_R	200
N_Q	1000
$\bar{\omega}_0$	$0_{7\times1}$
e_0	$0_{4\times1}$
R_{small}	10^{-10} I_4
R_{big}	10^{-2} I_4
$\tilde{Q}_{small}$	10^{-6} I_7
$\tilde{Q}_{big}$	I_7
Q_{small}	10^{-6} I_6
Q_{big}	I_6

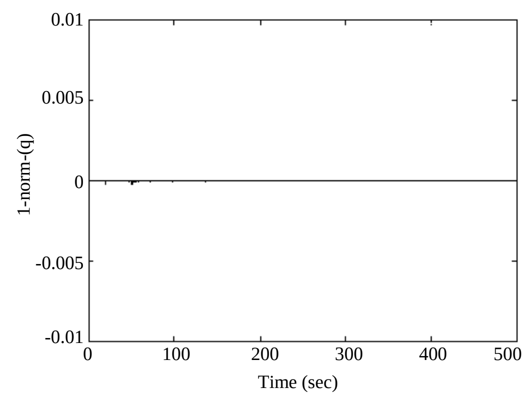

Fig. 1: Estimated quaternion norm error

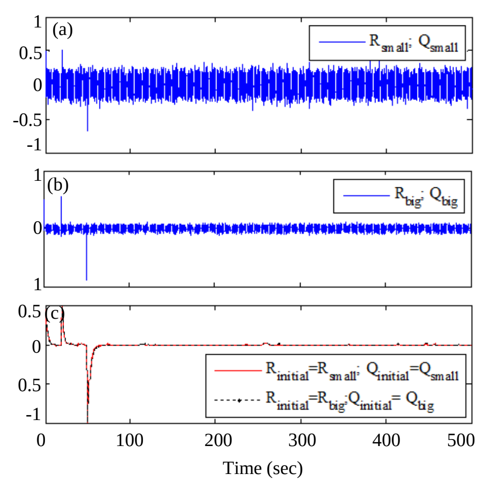

Fig. 2(a-c): Estimation error for b_1 with (a) EKF using R_{small} and Q_{small}, (b) EKF using Q_{big} and R_{big}, and (c) ACQEKF with covariance initialization as in the legend

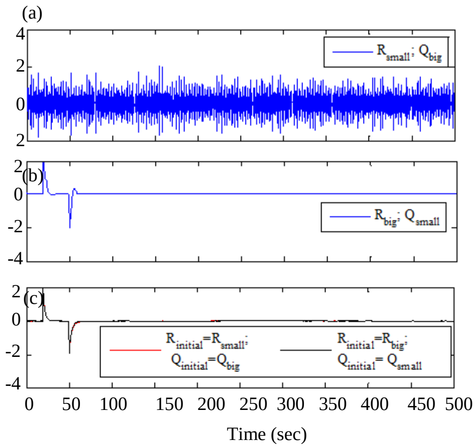

Fig. 3(a-c): Estimation error for b_2 with (a) EKF using R_{small} and Q_{big}, (b) EKF using Q_{small} and R_{big}, and (c) ACQEKF with covariance initialization as in the legend

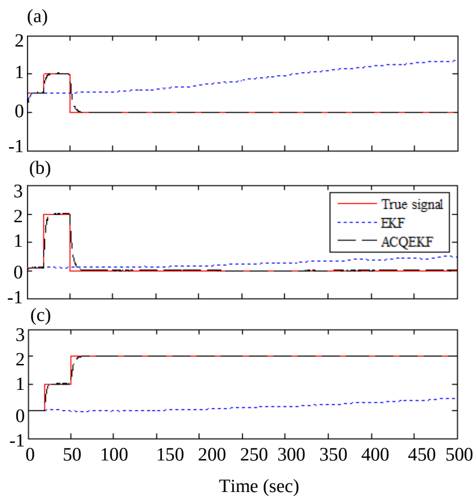

Fig. 4(a-c): b vector components, true (solid line), estimated from ACQEKF (dashed black line) and estimated from EKF (dotted blue line), (a) b_1, (b) b_2 and (c) b_3

The ACQEKF still superior for both cases as displayed in Fig. 3c. In the same context, a remarkable point for the ACQEKF is that it converges to the same result even though different covariance initializations are used.

Furthermore, some values for the covariances may slow the filter response or even cause divergence. For the selected values of $Q = 10^{-10}$ and $R = 10^{-2}$, the estimated bias using the EKF diverges, however, this problem is solved in the ACQEKF which keeps the stability of the filter and forces it to converge as shown in Fig. 4a. Moreover, the EKF converges very slowly while the ACQEKF still able to track the changes for the same values of covariances as depicted in Fig. 4b and c. This is because the EKF filter gain increases with increasing Q and decreasing R. If the selected value of Q is very small compared to Q_{True}, then the resulted gain is small. In terms of convergence, if $q \rightarrow 0$, then the value of $Y \rightarrow 1$ and thus the convergence of Eq. 80 is slowed down. However, the ACQEKF gain is changing based on the estimation performance. This is clear, since, it depends on both the innovation error e and the state error $\acute{\omega}$, respectively. If the tracking is not satisfied then the gain will not converge to very small value. The reason is that the value of the noise covariance will increase to change the gain for better performance. In the same context, increasing the noise covariance increases q which leads for better convergence as in Eq. 80. Thus, we can claim that the proposed ACQEKF has better stability and convergence performances than the EKF shows:

CONCLUSION

A new real time adaptive constrained quaternion Kalman filter ACQEKF for systems with uncertain noise covariances is proposed. This ACQEKF can adjust itself recursively to achieve better, more accurate and stable performance for biased covariances. It relates the filter gain to the innovation and state errors through the noise covariance updating rules, these relations change the filter gain for better tracking and performance. Furthermore, its tuning parameters are less than the EKF, instead of tuning all of the diagonal elements of the noise covariance matrix, they can be initialized and then tuned using and only. It also preserves the unity norm constraint for the quaternion during the running of the algorithm. The results show the dramatic improvements in the ACQEKF response compared with the conventional EKF under the same conditions.

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