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On the Prediction of Stable Equilibrium State of Linear Time-Invariant Multivariable Systems

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Abstract: A general representation for a class of linear time-invariant multivariable systems with interconnected scalar feedback loops was presented. The procedure was contrasted with the methods due to Liapunov, which predicted stability of equilibrium position directly without the explicit form of solution.

Key words: Stability, equilibrium state, liapunov function, time invariant, multivariable system

INTRODUCTION

The problems of stability in dynamical systems have been addressed largely in the existing literature (Likins, 1967; Walker, 1970), for example, many modern rotor systems are operated at high speeds and consequently several bending modes are excited. When the speed of the rotor system exceeds its critical value safe and reliable operational conditions can no longer be guaranteed. In the analysis of complex interconnected systems, overall stability is one of the most important considerations. A modern electric power plant, for example, is a complex multivariable feedback system with voltage- and frequency-dependent loads in which the prime objective is essentially the matching of generation to load demand. The plant generally operates under conditions of unsteady load. Both active and reactive power demands change continuously with rising or falling trend. When the load on the system is increased, the turbine speed drops thereby causing the governor to adjust the steam supply to turbo-generators or water jet supply to hydro-generators to the new load. As the change in the value of the speed diminishes, the generated error signal, which is transformed through an amplifier to the steam valve position command becomes smaller and the position of the governor flyballs gets closer (Weedy and Corby, 1970) to the point required to maintain a constant speed thereby creating an offset which must be overcome to restore the speed or frequency to its nominal value. It has been shown that turbine governors can be used to adjust power input to electricity generating plants in response to frequency deviation from the pre-set values. This is the only type of control required in isolated systems. Stability of electric power systems is concerned with characteristic behaviour of synchronous machines due to presence of disturbing forces; it is the ability of the machines to remain in operating equilibrium or in synchronism as long as disturbances last. In stable operations, the system develops restoring forces large enough to maintain state of equilibrium. If the forces tending to hold the machines in synchronism with one another are sufficient enough to overcome the disturbing forces, the system is said to be stable.

This study examines the stability of the equilibrium state of linear multivariable feedback systems in a typical plant operation and establishes criteria for stability of the system using both the explicit method and direct methods due to Liapunov.

THE DYNAMIC SYSTEM

The problem considered in this study is a linear time-invariant multivariable system, which by virtue of its generic nature finds ready application to modern power plants and similar multi-machine interconnections. The system is described mathematically as follows:

$$x\,(t) = Ax\,(t) + Bu(t) \qquad (1)$$

$$y\,(t) = Cx\,(t) \qquad (2)$$

Where $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{q\times n}$. $u\,(t) \in R^m$ is the input, $x\,(t) \in R^n$ is the state and $y\,(t) \in R^q$ the output. The system is denoted by (A, B, C) since they are completely determined by the matrices A, B and C. The zero-state input-output properties of this system are completely defined by its transfer function matrix H (s) specified as follows:

$$H(s) \triangleq C(sI-A)^{-1}B \qquad (3)$$

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where $H(s) \in R^{q\times m}$ whose elements are strictly proper rational functions in s. The problem is generally to find a triple (A, B, C) such that H (s) = C (sI − A) B and A are of least possible size and for any given q×m rational matrix $H_2(s)$, to obtain a state feedback law

$$u(t) = Gv(t) + Fx(t),\ G \in R^{m\times m} \text{ and } F \in R^{m\times n} \quad (4)$$

and an output feedback law

$$u(t) = Gv(t) + Ky(t),\ G \in R^{m\times m} \text{ and } K \in R^{m\times q} \quad (5)$$

such that the overall system transfer function matrix $C(sI - A - BF)^{-1}BG$ for the state feedback case and $C(sI - A - BKC)^{-1}BG$ for the output case, is exactly equal to the given rational matrix $H_2(s)$. This is basic in the design of multivariable feedback systems. The fundamental problem is to explore the analysis of linear multivariable feedback systems and establish sufficient conditions for stability of the systems. The approach was to describe the system by an interconnection of scalar blocks with inputs u_1, ..., u_m and outputs y_1, ..., y_m, defined mathematically as follows:

$$\begin{aligned} Y_i(s) &= g_i(s)v_i(s), \quad i = 1,...., m \\ v_i(s) &= \sum_{j=1}^{m} k_{ij}E_j(s), \quad i = 1,...., m \\ E_i(s) &= u_i(s) - y_i(s), i = 1,...., m \end{aligned} \quad (6)$$

Where v_i and e_i are auxiliary variables, i = 1, ..., m. k_{ij}, i = 1, ..., m are constants and $g_i(s)$, i = 1, ..., m are strictly proper rational transfer functions. To exclude trivial loops, we require no $g_i(s) = 0$ identically, i = 1, ..., m. A block diagram representation of (6) is shown in Fig. 1.

A stable variable realization of this system is easily obtainable. For matrices A_i, b_i and c_i^T, i = 1, ..., m, we consider the following:

Let

$$\begin{aligned} \frac{dx_i}{dt} &= A_i x_i + b_i v_i \\ y_i &= c_i^T, \quad i = 1, ..., m \end{aligned} \quad (7)$$

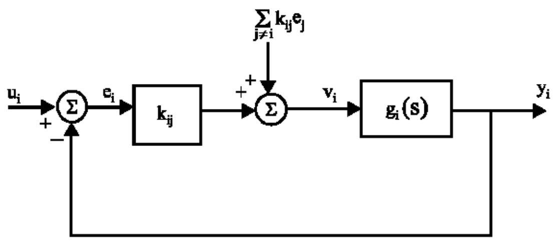

Fig. 1: A block diagram representation of $E_i(s) = u_i(s) - y_i(s)$, i = 1, ..., m

be a minimal realization of $g_i(s)$, i = 1, ..., m. Then

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{j=1}^{m}(A_i\delta_{ij} - b_i k_{ij} c_i^T) + \sum_{j=1}^{m} b_i k_{ij} u_j \\ y_i &= c_i^T x_i,\ i = 1, ..., m \end{aligned} \quad (8)$$

is minimal realisation of (6)

Where $\delta_{ij} = 1$ if $i = j$
$= 0$ if $i \neq j$

CONCEPT OF STABILITY

If x_1, x_2, ..., x_n represent n coordinates in an n-dimensional state space X and t the time, the behaviour of the system described by (1) and (2) is as follows:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, ..., x_n, t),\ i = 1, ..., m \quad (9)$$

or if x_i and f_i are the elements of the column vectors x and f, by the vector differential equation

$$\frac{dx}{dt} = f(x, t) \quad (10)$$

If a solution exists in some neighbourhood of a given initial condition (x_o, t_o) where $x_o \in X$, $t \in (-\infty, \infty)$ then it is denoted by $x(t; x_o, t_o)$. In the analysis of complex multivariable systems overall stability is one of the most important considerations. The equilibrium x = 0 of the model (1 and 2) is stable if for every real E>0 and t_o there exits a real $\delta(E, t_o)>0$ such that

$$\|x_o\| \le \delta \rightarrow \|x(t, x_o)\| \le E \quad (11)$$

For all $t \ge t_o$ (Obinabo, 1996). The system can be seen to satisfy these conditions globally since

$$\|x_o\| \le \infty \rightarrow \|x(t; x_o, t_o)\| = \|x_o\| \quad (12)$$

for all $x_o \ge t_o$

The equilibrium x = 0 of the system is attractive if for some ρ>0 and for every η>0 there exists a number $T(\eta, x_o, t_o)$ such that

$$\|x(t; x_o, t_o)\| \le \eta \quad (13)$$

for all $(t - t_o) \ge T$ and for all $||xo|| \le \rho$.

The equilibrium x = 0 of the system is asymptotically stable if it is both stable and attractive. In this study, for any given E < 0 and t_o, there exists a constant $\delta(E, t_o)$ such that

$$\left\|x(t; x_o, t_o)\right\| < E \tag{14}$$

For all $\|x_o\| \leq \delta$ and $\underset{t\to\infty}{Lt}\{x(t; x_o, t_o)\} = 0$

Theorem 1: The closed-loop system represented by (1) and (2) is stable if and only if all the zeros of

$$\det\left[\frac{\delta_{ij}}{g_i(s)} + k_{ij}\right]$$

have negative real parts. This result is valid if g_i (s) is proper and

$$\det\left[\delta_{ij} + g_i(\infty)k_{ij}\right] \neq 0$$

Proof: Let $\Delta_c(s)$ and $\Delta_o(s)$ denote the characteristic polynomials of the closed and open loop systems, respectively. Then

$$\begin{aligned}\Delta_c(s) &= \Delta_o(s)\det\left[\delta_{ij} + g_i(s)k_{ij}\right] \\ &= \Delta_o(s)\prod_{i=1}^{m} g_i(s)\det\left[\frac{\delta_{ij}}{g_i(s)} + k_{ij}\right]\end{aligned} \tag{15}$$

But the zeros of $\Delta_o(s)$ are the same as the poles of $\Pi g_i(s)$ and the zeros of $\Pi g_i(s)$ are the same as the poles of

$$\det\left[\frac{\delta_{ij}}{g_i(s)} + k_{ij}\right]$$

Hence, due to cancellations, the zeros of $\Delta_c(s)$ are the same as the zeros of

$$\det\left[\frac{\delta_{ij}}{g_i(s)} + k_{ij}\right]$$

Thus the closed-loop system is stable if and only if its zeros all have negative real parts.

Theorem 2: Suppose $\alpha_i < \beta_i$ and

$$\frac{1+\beta_i g_i(s)}{1+\alpha_i g_i(s)}$$

is positive real for i = 1, ..., m. The closed-loop system is stable if (α) there exists multipliers ρ>0, i = 1, .., m such that the matrix $\lfloor\lambda_{ij}\rfloor$ is positive definite where,

$$\begin{aligned}\lambda_{ii} &= \rho_i(\beta_i - k_{ii})(k_{ii} - \alpha_i) - \sum_{j=i}\rho_j k_{ji}^2 \\ \lambda_{ij} &= \lambda_{ji} = \frac{1}{2}\left\{\rho_i k_{ij}\left[(\beta_i - k_{ii}) - (k_{ii} - \alpha_i)\right]\right\} + \\ &\rho_j k_{ji}\left[(\beta_j - k_{jj}) - (k_{jj} - \alpha_j)\right] - \sum_{l=i}\rho_l k_{li} k_{ij}\end{aligned} \tag{16}$$

for i ≠ j. (b) The closed-loop system is stable if for some $0 \leq \theta \leq$,

$$\begin{aligned}\beta_i - k_{ii} &> \sum_{j=i}\left|\theta k_{ij} + (1-\theta)k_{ji}\right| \\ k_{ii} - \alpha_i &> \sum_{j=i}\left|\theta k_{ij} + (1-\theta)k_{ji}\right|\end{aligned} \tag{17}$$

for i = 1, ..., m.

Proof: After appropriate scalar transformations to each loop of (1) and (2) it can be shown that the zeros of

$$\det\left[\frac{\delta_{ij}}{g_i(s)} + k_{ij}\right]$$

are the same as the zeros of

$$\det\left[\rho_i\left(\frac{1+\alpha_i g_i(s)}{1+\beta_i g_i(s)}\right)\delta_{ij} + \rho_i\hat{k}_{ij}\right]$$

Where

$$\hat{k}_{ij},\ i, j = 1, ..., m$$

are the unique solutions of

$$\sum_{l=1}^{m}\left(\beta_i\delta_{il} - \hat{k}_{il}\right)k_{lj} = k_{ji} - \alpha_i\delta_{ij},\ i = 1, ..., m \tag{18}$$

Now suppose there is $\hat{s}$ with Re($\hat{s}$) > 0 such that.

$$\det\left[\rho_i\left(\frac{1+\alpha_i g(\hat{s})}{1+\beta_i g_i(\hat{s})}\right)\delta_{ij} + \rho_j\hat{k}_{ij}\right] = 0 \tag{19}$$

Thus the matrix

$$\left[\rho_i\left(\frac{1+\alpha_i g_i(\hat{s})}{1+\beta_i g_i(\hat{s})}\right)\delta_{ij} + \rho_j\hat{k}_{ij}\right]$$

is singular and there is non-zero (z_i, ..., z_m), with z_i complex such that

$$\rho_i\left(\frac{1+\alpha_i g_i(\hat{s})}{1+\beta_i g_i(\hat{s})}\right)z_i = -\sum_{j=1}^{m}\rho_j\hat{k}_{ij},\quad i = 1, ..., m \tag{20}$$

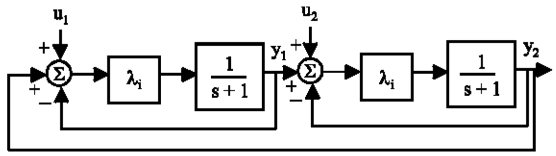

Fig. 2: Electric power system with minor and major feedbacks loops

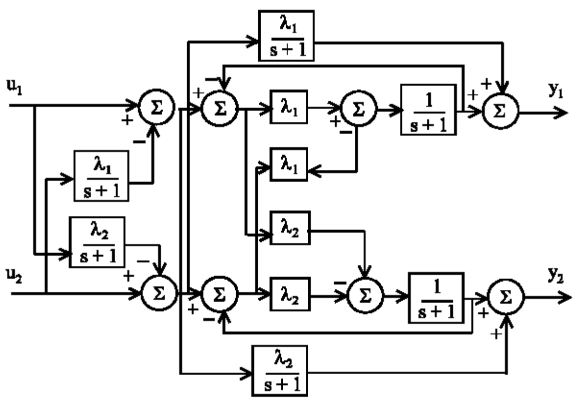

Fig. 3: The system is reduced to an equivalent representation

Consequently,

$$\sum_{i=1}^{m}\rho_i\left(\frac{1+\alpha_i g_i(\hat{s})}{1+\beta_i g_i(\hat{s})}\right)|z_i|^2 = -\sum_{i=1}^{m}\sum_{j=1}^{m}\bar{z}_i\rho_i\hat{k}_{ij}z_j \tag{21}$$

It can be shown that

$$\sum_i\sum_j \bar{z}_i\rho_i\hat{k}_{ij}z_j = \sum_i\sum_j \bar{z}_i\lambda_{ij}z_j \ > 0 \tag{22}$$

Where z_i, i = 1, ..., m are the unique solutions of

$$z_i = \sum_{j=1}^{m}\left(\beta_i\delta_{ij} - k_{ij}\right)z_j,\ i = 1, ..., m \tag{23}$$

Thus

$$\sum_{i=1}^{m}\rho_i\left(\frac{1+\alpha_j g_i(\hat{s})}{1+\beta_i g_i(\hat{s})}\right)|z_i|^2 \ < 0 \tag{24}$$

and hence for some j

$$\mathrm{Re}\left\{\frac{1+\beta_i g_i(\hat{s})}{1+\alpha_i g_i(\hat{s})}\right\} < 0 \tag{25}$$

a contradiction.

Proof:
Since

$$\mathrm{Re}\left\{\frac{1+\beta_i g_i(s)}{1+\alpha_i g_i(s)}\right\} \geq 0 \tag{26}$$

If $\mathrm{Re}\{s\} \geq 0$ i = 1, ..., m, the inequalities can be seen to guarantee that

$$\left|\frac{1}{g_i(s)} + k_{ii}\right| > \sum_{j\neq i}\left|\theta k_{ij} + (1-\theta)k_{ji}\right| \tag{27}$$

$$\left|\frac{1}{g_i(s)} + \frac{\alpha_i+\beta_i}{2}\right| \geq \left|\frac{\beta_i-\alpha_i}{2}\right| \tag{28}$$

Example: Consider a two-input, two-output sub-unit interconnection of two generators of an electric power system with minor and major feedbacks loops in Fig. 2.

For ease of application of the above theorem the system is reduced to an equivalent representation as in Fig. 3.

We observe that the sub-unit with the feedback loops is of the form shown in (1), with

$$g_1(s) = g_2(s) = \frac{1}{s+1} \tag{28}$$

and $k_{11} = \lambda_1$, $k_{12} = -\lambda_1$, $k_{21} = -\lambda_2$, $k_{22} = \lambda_2$
Using the theorem, the closed loop is stable if

$$\lambda_1 > -1,\ \lambda_2 > -1$$

$$(\lambda_1+1)(\lambda_2+1) > \frac{1}{4}(\lambda_1+\lambda_2)^2$$

Also with

$$\theta = \frac{1}{2}$$

the closed-loop is stable with

$$\min\{1+\lambda_1, 1+\lambda_2\} > \left|\frac{\lambda_1+\lambda_2}{2}\right|$$

or, with $\theta = 1$, if $-\frac{1}{2} < \lambda_1,\ -\frac{1}{2} < \lambda_2$

The concept due to Liapunov is embodied in the reasoning that if the rate of change

$$\frac{dE(x)}{dt}$$

of the energy E(x) of a physical system is negative for every possible state x except for a single equilibrium state

x_e, then the energy will continually decrease until it finally assumes its minimum value $E(x_e)$. The function E(x) is said to be a Liapunov function in the region R the state space provided that:

- It is positive-definite in R.
- Its time derivative with respect to the system (1) and (2) is continuous and negative semi-definite in R. The function E(x, t) is said to be a Liapunov function in the region ϕ of the solution space (X, x, t) provided that

- It is positive-definite in ϕ
- Its time derivative with respect to the system

  $$\frac{dE}{dt} = \sum_{i=1}^{n} \frac{\partial E}{\partial x_i} + \frac{\partial E}{\partial t}$$

  is continuous and negative semi-definite in ϕ.

If the Liapunov function E(x, t) with continuous first partial derivatives satisfies the following conditions

- $E(0, t) = 0$ for all t
- $E(x, t) > 0$ for all $x \neq 0$ and for all t.
- $E(0,t) = 0$ for all t
- $E(x, t) < 0$ for all $x \neq 0$ and for all t
- $E(x,t) \to \infty$ as $\|x\| \to$ for all t.

then the origin is globally asymptotically stable. These conditions have been established (Obinabo, 1996) for the equilibrium condition of material flow in iron-ore pelletisation process and satisfy the general problem description (1) and (2) thereby confirming the Liapunov property for the multivariable feedback system.

CONCLUSION

A sufficient and valid condition for multivariable feedback systems to be stable has been established. The stability condition of a continuous-time composite system have been given in terms of constants characterizing the sub-systems and their interconnection (Bailey, 1966). An application of the vector Liapunov function method assumed that for all isolated subsystems there exist Liapunov functions of second order and that the interconnecting relations are linear and time-invariant (Thompson, 1970). The application gives a sufficient condition for asymptotic stability in the large as a set of defined inequalities. The results established in this study can be compared with conditions for stability of multivariable systems reported in the existing literature.

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