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On Certain Propertis of Bond Prices

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Abstract: The study offers a theorem on the properties of the bond price; it is formulated and proved mathematically-under the same conditions as established theorems on the bond valuation. The theorem concerns the previously unstudied dependency of bond price variation at changing terms to maturity on the level of required yield. For bonds that are sold at a discount the maximums of the absolute and relative bond price changes are found. The assertions proved in the theorem are confirmed by the proper calculations.

Key words: Bond price properties, mathematical methods, the influence of the market profitability level, terms, sold

INTRODUCTION

There are five known theorems on the bond valuation (Malkiel, 1962). These are the theorems about the properties of the bond price. The theorems have been formulated resulting from the observations of the bond market. In the work (Malkiel, 1962), they are formulated and proved as mathematical theorems. In recent decades, the application of mathematical methods in the investment analysis has been recognized as effective and necessary. There are other proofs of these theorems (available, for example in (Barbaumov *et al*., 2006; Melnikov *et al*., 2006; Lawrence and Shankar, 2007)) and their generalization (Barber, 2010). Properties of bond price formulated in these theorems are "important to predict the effect of interest rate on bonds rates" (Sharpe *et al*., 1999). Currently, these theorems are the base for the theory of financial investment with a fixed income under conditions of certainty. Despite the serious limitations of this theory, its results are required as a part of the general theory of investment. This is assured, for example, by the research of Geoffrey Poitras Frederick.

The present study provides a proof of another theorem about the properties of the bond price. This is a theorem concerning the effect of the required yield level on the change in bond price under changing terms to maturity. The required yield or simply yield refers to the yield to maturity (Fabozzi, 2000). This dependence has not been previously considered in the existing literature on investments. The theorem is stated and proved under the same conditions as the established theorems from the work (Malkiel, 1962). These are the conditions of certainty. The basic requirements of certainty conditions are the following: the bond is fairly valued. Sharpe *et al*. (1999) has no credit risk and cannot be revoked by the issuer prior to the stated maturity date. To prove the theorem, we used the properties of differentiable functions and numerical sequences. The theorem is proved for coupon bonds.

Let us consider a bond with a par value A. Coupon disbursements against this bond are made once a year at an annual coupon rate f. Let to be the bond price at the time, when n coupon periods remain before its maturity and r to be the bond yield at this time point. Then:

$$P_n = \sum_{i=1}^{n} \frac{q}{(1+r)^i} + \frac{A}{(1+r)^n} \quad (1)$$

where q = fA is the coupon payment on the bond. Let us assume that, at the level of required yield r, the term to bond maturity has decreased from n to (n-1) coupon periods, where n>1. Then, the price of a bond sold at a premium, will be reduced by the amount of $\Delta p_n = P_n - P_{n-1}$ while the price of a bond sold at a discount will be rised by the amount of $\Delta p_n = P_{n-1}-P_n$. For the bonds sold at their par values, $\Delta p_n = 0$. Thus, the change in the price of a bond sold at a premium or discount, when decreasing the term to maturity by one coupon period, is positive by definition. This is an absolute change in the bond price taken with the "+" sign. The absolute ΔP_n and relative $\Delta P_n/P_n$ values of bond price change, when reducing the term to maturity by one coupon period will be regarded as a function of the yield r. Please, note that one of the known theorems from (Malkiel, 1962) (theorem No. 2) considers the dependence of ΔP_n on n at a fixed r.

Theorem: At a fixed n>1 and f, the following assertions are true:

- Δp_n and $\Delta P_n/P_n$ and are decreasing functions on the segment $0 \le r \le f$ (the bond is sold at a premium when r<f and at par when r = f)
- There are point maximums for the function and for the function (for) on the set (the bond is sold at a discount when)

Theorem proving: Let prove the assertions of the theorem for the absolute change in the bond price. For a bond sold at a premium (r<f), the function ΔP_n is continuous within a closed segment [0, f] and differentiable on the interval (0, f). Since, the derivative $(\Delta P_n)'_r = (P_n - P_{n-1})'_r < 0$ on the set 0<r<f, then the function is decreasing on the segment $0 \le r \le f$ and $\Delta p_n(r = 0) = q$, $\Delta p_n(r = f) = 0$.

For a bond sold at a discount (r>f), the function ΔP_n is continuous on the semi-interval [f, +∞) and differentiable on the interval (f, +∞). Then:

$$(\Delta P_n)'_r = (P_{n-1} - P_n)'_r = \begin{cases} >0, & f < r < r_a \\ <0, & r > r_a \end{cases} \quad (2)$$

Where:

$$r_a = \frac{1+nf}{n-1} \quad (3)$$

Therefore, r_a is the maximum of the function ΔP_n on the semi-interval [f, +∞) and $\Delta p_n(r = f) = 0$:

$$\Delta P_n(r_a) = A\frac{r_a - f}{(1+r_a)^n} > 0, \lim_{r\to+\infty} \Delta P_n(r) = 0$$

Thus, the assertions of the theorem for the absolute changes of price ΔP_n are proven. Let us prove the assertions of the theorem for the relative change of the bond price. Let r<f. We consider the derivative of the function $\Delta P_n/P_n$ with respect to r, where $\Delta P_n = P_n\text{-}P_{n-1}$. We obtain:

$$\left(\frac{\Delta P_n}{P_n}\right)'_r = \left(1 - \frac{P_{n-1}}{P_n}\right)'_r = -\left(\frac{P_{n-1}}{P_n}\right)'_r = \frac{P_{n-1}}{P_n}\left(\frac{P'_n}{P_n} - \frac{P'_{n-1}}{P_{n-1}}\right) \quad (4)$$

We use the ratios:

$$\frac{P'_n}{P_n} = -D_n(r)\frac{1}{1+r}$$

And:

$$\frac{P'_{n-1}}{P_{n-1}} = -D_{n-1}(r)\frac{1}{1+r}$$

where $D_n(r)$ and $D_{n-1}(r)$ are the Macaulay duration of the bond at the terms to maturity with *n* and coupon periods and the bond yield r. Then:

$$\left(\frac{\Delta P_n}{P_n}\right)'_r = \frac{P_{n-1}}{P_n(1+r)}(D_{n-1}(r) - D_n(r)) \quad (5)$$

It was found (Barbaumov *et al*., 2006; Melnikov *et al*., 2006) that at th bond yield r<f, the sequence $\{D_n(r)\}$ is increasing. Then $(\Delta P_n/P_n)'_r<0$ on the interval (0, f). Therefore, the ratio $\Delta P_n/P_n$ is a decreasing function with respect to bond yield r on the segment [0, f] and:

$$\frac{\Delta P_n}{P_n}(r=0) = \frac{q}{A+nq}, \ \frac{\Delta P_n}{P_n}(r=f) = 0$$

Thus, the assertion of the theorem for a bond sold at a premium (r<f), is proven. Let r>f. Differentiating the function $\Delta P_n/P_n$ with respect to r, where $\Delta P_n = P_{n-1}\text{-}P_n$, we obtain:

$$\left(\frac{\Delta P_n}{P_n}\right)'_r = \frac{P_{n-1}}{P_n(1+r)}(D_n(r) - D_{n-1}(r)) \quad (6)$$

In the study (Popova, 2011), it has been revealed that for a bond sold at a discount with the bond yield r>f, there exists a term $n_o = n_o(r)$ such that for the terms to maturity $n<n_o$ the sequence $\{D_n(r)\}$ is increasing while for $n>n_o$ the sequence $\{D_n(r)\}$ is decreasing. Under the condition n>1, we obtained an approximate value of the term n_o:

$$n_o \approx \frac{1}{r} + \frac{1+r}{r-f} \quad (7)$$

Consequently, $n_o>2$. Figure 1 shows the dependence of the term n_o on r according to the Eq. 2. Then for the fixed n>2 we obtain:

$$\left(\frac{\Delta P_n}{P_n}\right)'_r = \begin{cases} >0, & f < r < r_o \\ <0, & r > r_o \end{cases} \quad (8)$$

where r_o is the solution of the equation $n_o(r) = n$ (Fig. 1). It follows that r_o is the maximum point of the function $\Delta P_n/P_n$ on the set [f, +∞) and:

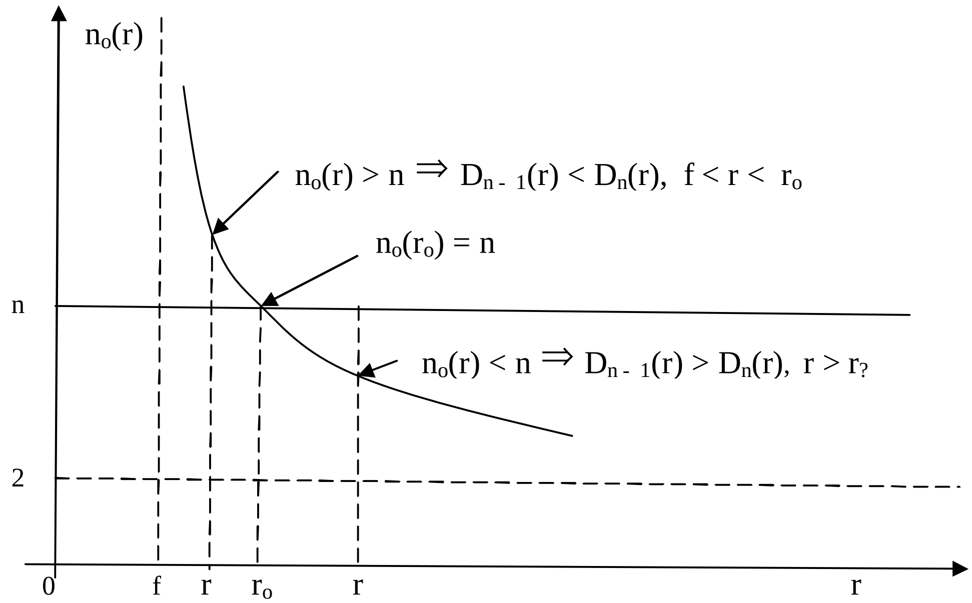

Fig. 1: The dependence of the term n_o on the bond yield r

Table 1: A = 100, f = 10%, n = 10

r (%)	ΔP_n	$\Delta P_n/P_n$
1	8.147583	0.04398351
2	6.562786	0.03818667
3	5.208657	0.03261293
8	0.926387	0.00816775
9	0.422411	0.00396937
10	0.000000	0.00000000
11	0.352184	0.00374223
12	0.643946	0.00725986
13	0.883765	0.01055604
22.1	1.642967	0.03118583
22.2	1.643039	0.03132813
22.3	1.643011	0.03146879
34.0	1.285770	0.03873566
34.1	1.281531	0.03873607
34.2	1.277292	0.03873571
34.5	1.264571	0.03873002
35	1.243376	0.03870559
35.5	1.222214	0.03866313

Table 2: A = 100, f = 10%

	Percentage					
n	10	20	30	40	50	100
r_a	22.20	15.70	13.80	12.80	12.20	11.10
r_a (1)	22.22	15.79	13.79	12.82	12.24	11.11
r_o	34.10	18.80	15.20	13.60	12.70	11.20
r_o (3)	29.58	18.15	14.93	13.48	12.68	11.22

$$\frac{\Delta P_n}{P_n}(r=f)=0,$$
$$\frac{\Delta P_n}{P_n}(r_o)=\frac{r_o(r_o-f)}{(r_o-f)+f(1+r_o)^n}>0, \qquad (9)$$
$$\lim_{r\to+\infty}\frac{\Delta P_n}{P_n}=0$$

For $n_o(r)$ from Eq. 2, the approximate solution of the equation $n_o(r)$ has the following form:

$$r_o \approx \frac{(nf+2)+\sqrt{n^2f^2+4+4f}}{2(n-1)} \qquad (10)$$

Thus, the assertion of the theorem for a bond sold at a discount is proven. Therefore, the theorem is proved. Table 1 shows the calculations of the absolute ΔP_n and relative $\Delta P_n/P_n$ changes of the bond price, when reducing the term to maturity by one cupon period for a fixed n at different values of $r \leq f$ and $r > f$. The rows in the Table 1, which correspond to the maximum points r_a and r_o in the field of $r > f$, are highlited. It is obvious from Table 1 that the calculation results confirm the proven assertions 1 and 2 of the theorem.

Table 2 presents the values of the maximum points r_a and r_o over the range $r > f$ and various n. The values of r_a and r_o obtained after direct computation of ΔP_n and $\Delta P_n/P_n$ for different r values are compared with the calculations according to the Eq. 1 and 3, respectively (with the increase of n, the value of r_o calculated by Eq. 3 becomes more precise).

RESULTS AND DISCUSSION

The main result of the present article is the mathematical proof of the theorem and its confirmation by the calculations. The theorem establishes the influence of the required yield level on the change in the bond price when reducing the term to maturity by one coupon period. For bonds that are sold at a discount the existing maxima of the ΔP_n and $\Delta P_n/P_n$ functions has been proved mathematically and verified by calculated data.

The performed calculations (Table 2) and the presented Fig. 1 shows that the yields r_a and r_0 which are close to the value of coupon rate, correspond to higher values of n.

CONCLUSION

From this, we can conclude that the results of the theorem can be useful when investing in long-term bonds.

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